A completion construction for continuous dynamical systems *†‡

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March 1, 2012

Abstract In this work we construct the $\check{C}_0^{\mathbf{r}}$-completion and $\check{C}_0^{\mathbf{l}}$-completion of a dynamical system. If X is a flow, we construct canonical maps $X \to \check{C}_0^{\mathbf{r}}(X)$ and $X \to \check{C}_0^{\mathbf{l}}(X)$ and when these maps are homeomorphism we have the class of $\check{C}_0^{\mathbf{r}}$-complete and $\check{C}_0^{\mathbf{l}}$-complete flows, respectively. In this study we find out many relations between the topological properties of the completions and the dynamical properties of a given flow. In the case of a complete flow this gives interesting relations between the topological properties (separability properties, compactness, convergence of nets, etc.) and dynamical properties (periodic points, omega limits, attractors, repulsors, etc.).

1 Introduction

Some origins of dynamical system and flow theory can be established with the pioneering work of H. Poincaré [20, 21] in the late XIX century studied the topological properties of solutions of autonomous ordinary differential equations. We can also mention the work of A. M. Lyapunov [16] who developed his theory of stability of a motion (solution) of a system of n first order ordinary differential equations. While much of Poincaré's work analyzed the global properties of the system, Lyapunov's work looks at the local stability of a dynamical system. The theory of dynamical systems reached a great development with the work of G.D. Birkhoff [2], who may be considered as the founder of this theory. He

*2010 Mathematics Subject Classification: 18B99, 18A40, 37B99, 54H20.
†Key words and phrases: Dynamical system, exterior space, exterior flow, limit flow, end flow, completion flow.
‡The authors acknowledge the financial support provided by Ministerio de Educación y Ciencia, grant MTM2009-12081, FEDER and the University of La Rioja, EGI11/55.

established two main lines in the study of dynamical systems: the topological theory and the ergodic theory.

This paper develops some new ideas in the topological theory of dynamical systems based in the theory of exterior spaces. A new construction "the $\check{C}_0$-completion of a flow" has been introduced in this work. On the one hand, using this construction we can clarify the interrelations between some topological properties of the $\check{C}_0$-completion of a flow (separation, local compactness, local path-connectness, etc) and dymamical properties of the flow (relations between critical points, periodic points, omega limits, attractors, etc). On the other hand, many known results of the class of T_2 compact flows in which all periodic points are critical can be applied to the completion of a flow X to obtain nice dynamical properties of the original flow X.

Previously, the authors have developed some results on proper homotopy theory and exterior homotopy theory to classify non compact spaces and to study the shape of a compact metric space, see [5, 9, 11]. Nevertheless, in this work, our main objective is to construct a completion of a flow using exterior spaces.

Firstly, for an exterior space X we can construct the limit space $L(X)$ and the end space $\check{\pi}_0(X)$. Next, the limit space and the end space are used to construct the completion $\check{C}_0(X)$. This construction can be considered as a generalization of the Freudenthal compactification given in [7].

Secondly, we introduce a hybrid structure called exterior flow that mixes the notion of dynamical system and exterior space. In particular, we can consider the limit space, the end space and the completion of an exterior flow. In the approach given in this paper, the main key to establish a connection from dynamical systems to exterior flows is the notion of absorbing open region (or $\mathbf{r}$-exterior subset). Given a flow on space X, an open set E is said to be $\mathbf{r}$-exterior if for every $x \in X$ there is $r_0 \in \mathbb{R}$ such that $r \cdot x \in E$ for $r \geq r_0$. The space X together with the family of $\mathbf{r}$-exterior open subsets is an exterior flow, which is denoted by $X^{\mathbf{r}}$.

Finally, we can give the limit space, the end space and the completion of a flow X by applying these constructions, already developed for exterior spaces, to the exterior flow $X^{\mathbf{r}}$.

An exterior space X is said $\check{C}_0$-complete if the natural transformation $X \to \check{C}_0(X)$ is an isomorphism. For a $\check{C}_0$-complete exterior space X the limit space $L(X)$ is isomorphic to the end space $\check{\pi}_0(X)$ and there is a class of nets (π_0-$\varepsilon(X)$-nets) which has at least a limit point in $L(X)$. Moreover, the limit space $L(X)$ can be considered as a global "weak attractor" of the exterior space X.

In section 3 we have constructed and studied the completion of an exterior space. We have given some characterizations, see Theorems 3.2, 3.4, of $\check{C}_0$-completeness for general exterior spaces and for locally path-connected exterior spaces. We have also proved that if X is a locally path-connected exterior space which is first countable at infinity, then $\check{C}_0(X)$ is $\check{C}_0$-complete.

An interesting class of Hausdorff compact $\check{C}_0$-complete spaces is given in Theorem 3.5. We have proved that if X is a Hausdorff compact space and D is a closed totally disconnected subspace (a Stone space), the externology $\varepsilon(X)$ of

open neighborhoods of D determines a $\check{C}_0$-complete exterior space.

In Theorem 3.6 we have given conditions to ensure that the limit space $L(\check{C}_0(X))$ and the completion $\check{C}_0(X)$ are Hausdorff spaces. Under the conditions given in Theorem 3.7, one has that $L(\check{C}_0(X))$ and $\check{C}_0(X)$ are compact exterior spaces.

We have proved the following result, which is a generalization of the Freudenthal compactification:

Theorem 3.8 Suppose that X is a locally path-connected, connected, Hausdorff exterior space such that the complement of a exterior open subset is compact. If for every $x \in X \setminus L(X)$ there is a closed neighborhood F such that $X \setminus F$ is exterior, then $\check{C}_0(X)$ is a Hausdorff compact exterior space and $L(X)$ is a closed subspace.

The completions construction developed for exterior spaces can be applied to **r**-exterior flows. In section 5 we have proved that the $\check{C}_0$-completion of an **r**-exterior flow has a canonical induced structure of **r**-exterior flow, moreover, this $\check{C}_0$-completion has as limit flow its set of critical points which also is a minimal weak attractor.

As a consequence of these facts, the class of $\check{C}_0$-complete **r**-exterior flows has very interesting properties: The limit flow agrees with the end flow which is the set of critical points and this limit is a global weak attractor. When, in addition, the **r**-exterior flow is locally path-connected, this limit subflow has the topology of a Hausdorff totally disconnected space. If a $\check{C}_0$-complete **r**-exterior flow is also T_2, we have that the limit space agrees with the omega limit of all the flow which, under these conditions, is the unique minimal global attractor. We remark that for the class of Hausdorff compact $\check{C}_0$-complete **r**-exterior flows we have the additional property that the limit has the topology of a profinite space.

In section 6, we consider two full embeddings of the category of flows to the category of **r**-exterior flows denoted by $X \to X^{\mathbf{r}}$ and $X \to X^{\mathbf{l}}$. This permits to construct the $\check{C}_0^{\mathbf{r}}$-completion of a flow and the $\check{C}_0^{\mathbf{l}}$-completion and the corresponding classes of $\check{C}_0^{\mathbf{r}}$-complete and $\check{C}_0^{\mathbf{l}}$-complete flows. These classes of flows have nice properties; for instance, the limit of a Hausdorff compact $\check{C}_0^{\mathbf{r}}$-complete ($\check{C}_0^{\mathbf{l}}$-complete) flow is the closed subset of critical points which agree with the set of periodic points and it is the minimal global attractor (repulsor). Moreover, in this case, this limit is a Stone space. For example, for compact metrizable flows, we have proved the following result:

Theorem 6.6 Let X be a locally path-connected, compact metric flow. Then, X is a $\check{C}_0^{\mathbf{r}}$-complete flow if and only if $C(X) = \overline{\Omega^{\mathbf{r}}(X)}$ and $C(X)$ is a Stone space .

Here $C(X)$ is the set of critical points and $\Omega^{\mathbf{r}}(X)$ is the global $\omega^{\mathbf{r}}$-limit of X. Notation and definitions can be seen in section 2.2.

Note that for a flow X with non-empty $\omega^{\mathbf{r}}$-limits, $\overline{\Omega^{\mathbf{r}}(X)}$ is the minimal closed global attractor. Therefore, in the context of Theorem 6.6, the set of critical points of the flow is a minimal closed global attractor.

We point out that a slight modification of the externologies considered in this paper taking as new **R**-exterior open subsets those open subsets E such

that for every $x \in X$, there are $r \in \mathbb{R}$ and U an open neighbourhood at x with $(r, \infty) \cdot U \subset E$, will permit to introduce new limit flows, end flows and completions. The techniques developed in this paper together with the new constructions will allow us to analyze stability properties of global attractors.

We remark that one of the tools used to study the topology of (stable) attractors is the shape theory and their associated algebraic invariants, see for instance [8], [18],[19],[23],[24]. We emphasize the fact that for flows on ANR spaces, the externologies considered in this paper are resolutions of the limit in the sense of shape theory. Therefore, many of the results obtained in all recent developments on the study of the properties of (un)stable attractors by means of shape theory can be related to other analogues given via the theory of exterior spaces.

2 Preliminaires on exterior spaces and dynamical systems

2.1 The category of proper and exterior spaces

A continuous map $f : X \to Y$ is said to be proper if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of topological spaces and the subcategory of spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera. An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps, see [9, 11]. For more properties and applications of exterior and proper homotopy categories we refer the reader to [10, 5, 3, 12, 13] and for a survey of proper homotopy to [22] .

Definition 2.1. *Let $(X, \mathbf{t})$ be a topological space, where X is the subjacent set and $\mathbf{t}$ its topology. An* externology *on $(X, \mathbf{t})$ is a non empty collection ε (also denoted by $\varepsilon(X)$) of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \mathbf{t}$ and $E \subset U$ then $U \in \varepsilon$. The members of ε are called exterior open subsets. A subfamily $\mathcal{B} \subset \varepsilon$ is said to be a base for ε if for every $E \in \varepsilon$ there is $B \in \mathcal{B}$ such that $B \subset E$.*

An exterior space *$(X, \varepsilon, \mathbf{t})$ consists of a space $(X, \mathbf{t})$ together with an externology ε.*

A map $f : (X, \varepsilon, \mathbf{t}) \to (X', \varepsilon', \mathbf{t}')$ is said to be an exterior map *if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

The category of exterior spaces and exterior maps will be denoted by **E**. Given a space $(X, \mathbf{t}_X)$, we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ or the total exterior space if one takes $\varepsilon = \mathbf{t}_X$. In this paper, we

shall use the exterior space of real numbers $(\mathbb{R}, \mathbf{r})$, where $\mathbf{r}$ is the externology determined by the externology base $\{(n, +\infty) | n \in \mathbb{Z}\}$. An important example of externology on a given topological space X is the one constituted by the complements of all closed-compact subsets of X, that will be called the cocompact externology and usually written as $\varepsilon^{\mathbf{c}}(X)$. The category of spaces and proper maps can be considered as a full subcategory of the category of exterior spaces via the full embedding $(\cdot)^{\mathbf{c}} : \mathbf{P} \hookrightarrow \mathbf{E}$. The functor $(\cdot)^{\mathbf{c}}$ carries a space X to the exterior space $X^{\mathbf{c}}$ which is provided with the topology of X and the externology $\varepsilon^{\mathbf{c}}(X)$. A map $f \colon X \to Y$ is carried to the exterior map $f^{\mathbf{c}} \colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ given by $f^{\mathbf{c}} = f$. It is easy to check that a continuous map $f : X \to Y$ is proper if and only if $f = f^{\mathbf{c}} \colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ is exterior.

An important role in this paper will be played by the following construction $(\cdot)\bar{\times}(\cdot)$: Let $(X, \varepsilon(X), \mathbf{t}_X)$ be an exterior space, $(Y, \mathbf{t}_Y)$ a topological space and for $y \in Y$ we denote by $(\mathbf{t}_Y)_y$ the family of open neighborhoods of Y at y. We consider on $X \times Y$ the product topology $\mathbf{t}_{X \times Y}$ and the externology $\varepsilon(X \bar{\times} Y)$ given by those $E \in \mathbf{t}_{X \times Y}$ such that for each $y \in Y$ there exists $U_y \in (\mathbf{t}_Y)_y$ and $T^y \in \varepsilon(X)$ such that $T^y \times U_y \subset E$. This exterior space will be denoted by $X \bar{\times} Y$ in order to avoid a possible confusion with the natural product externology. This construction gives a functor

$$(\cdot)\bar{\times}(\cdot) \colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}.$$

When Y is a compact space, we have that E is an exterior open subset of $X \bar{\times} Y$ if and only if it is an open subset and there exists $G \in \varepsilon(X)$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon(X) = \varepsilon^{\mathbf{c}}(X)$, then $\varepsilon(X \bar{\times} Y)$ coincides with $\varepsilon^{\mathbf{c}}(X \times Y)$ the externology of the complements of closed-compact subsets of $X \times Y$. We also note that if Y is a discrete space, then E is an exterior open subset of $X \bar{\times} Y$ if and only if it is open and for each $y \in Y$ there is $T^y \in \varepsilon(X)$ such that $T^y \times \{y\} \subset E$.

This bar construction provides a natural way to define *exterior homotopy* in $\mathbf{E}$. Indeed, if $I = [0, 1]$ denotes the closed unit interval, given exterior maps $f, g : X \to Y$, it is said that f *is exterior homotopic to* g if there exists an exterior map $H : X \bar{\times} I \to Y$ (called exterior homotopy) such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for all $x \in X$. The corresponding homotopy category of exterior spaces will be denoted by $\pi\mathbf{E}$. Similarly, the usual homotopy category of topological spaces will be denoted by $\pi\mathbf{Top}$.

2.2 Dynamical Systems and Ω-Limits

Next we recall some elementary concepts about dynamical systems; for more complete descriptions and properties we refer the reader to [1].

Definition 2.2. *A* dynamical system *(or a* flow*) on a topological space* X *is a continuous map* $\varphi \colon \mathbb{R} \times X \to X$, $\varphi(t, p) = t \cdot p$, *such that*

(i) $\varphi(0, p) = p$, $\forall p \in X$;

(ii) $\varphi(t, \varphi(s, p)) = \varphi(t + s, p)$, $\forall p \in X$, $\forall t, s \in \mathbb{R}$.

A flow on X will be denoted by (X, φ) and when no confusion is possible, we use X for short.

For a subset $A \subset X$, we denote $\mathrm{inv}(A) = \{p \in A | \mathbb{R} \cdot p \subset A\}$.

Definition 2.3. *A subset S of a flow X is said to be* invariant *if $inv(S) = S$.*

Given a flow $\varphi\colon \mathbb{R} \times X \to X$ one has a subgroup $\{\varphi_t\colon X \to X | t \in \mathbb{R}\}$ of homeomorphisms, $\varphi_t(p) = \varphi(t, p)$, and a family of motions $\{\varphi^p\colon \mathbb{R} \to X | p \in X\}$, $\varphi^p(t) = \varphi(t, p)$.

Definition 2.4. *Given two flows $\varphi\colon \mathbb{R} \times X \to X$, $\psi\colon \mathbb{R} \times Y \to Y$, a* flow morphism *$f\colon (X, \varphi) \to (Y, \psi)$ is a continuous map $f\colon X \to Y$ such that $f(r \cdot p) = r \cdot f(p)$ for every $r \in \mathbb{R}$ and for every $p \in X$.*

We note that if $S \subset X$ is invariant, S has a flow structure and the inclusion is a flow morphism.

We denote by $\mathbf{F}$ the category of flows and flows morphisms.

We recall some basic fundamental examples: (1) $X = \mathbb{R}$ with the action $\varphi\colon \mathbb{R} \times X \to X$, $\varphi(r, s) = r + s$. (2) $X = S^1 = \{z \in \mathbb{C} | |z| = 1\}$ with $\varphi\colon \mathbb{R} \times X \to X$, $\varphi(r, z) = e^{2\pi i r} z$. (3) $X = \{0\}$ with the trivial action $\varphi\colon \mathbb{R} \times X \to X$ given by $\varphi(r, 0) = 0$. In all these cases, the flows have only one trajectory.

Definition 2.5. *For a flow X, the* $\omega^{\mathbf{r}}$-limit set *(or right-limit set, or positive limit set) of a point $p \in X$ is given as follows:*

$$\omega^{\mathbf{r}}(p) = \{q \in X | \exists \text{ a net } t_\delta \to +\infty \text{ such that } t_\delta \cdot p \to q\}.$$

If $\overline{A}$ denotes the closure of a subset A of a topological space, we note that the subset $\omega^{\mathbf{r}}(p)$ admits the alternative definition:

$$\omega^{\mathbf{r}}(p) = \bigcap_{t \geq 0} \overline{[t, +\infty) \cdot p}$$

which has the advantage of showing that $\omega^{\mathbf{r}}(p)$ is closed.

Definition 2.6. *The* $\Omega^{\mathbf{r}}$-limit set *of a flow X is the following invariant subset:*

$$\Omega^{\mathbf{r}}(X) = \bigcup_{p \in X} \omega^{\mathbf{r}}(p)$$

Now we introduce the basic notions of critical, periodic and **r**-Poisson stable points.

Definition 2.7. *Let X be a flow. A point $x \in X$ is said to be a* critical point *(or a* rest point*, or an* equilibrium point*) if for every $r \in \mathbb{R}$, $r \cdot x = x$. We denote by $C(X)$ the invariant subset of critical points of X.*

Definition 2.8. *Let X be a flow. A point $x \in X$ is said to be* periodic *if there is $r \in \mathbb{R}$, $r \neq 0$ such that $r \cdot x = x$. We denote by $P(X)$ the invariant subset of periodic points of X.*

It is clear that a critical point is a periodic point. Then

$$C(X) \subset P(X).$$

If $x \in X$ is a periodic point but not critical, then there exists a real number $r \neq 0$ such that $r \cdot x = x$ and r is called a *period* of x. The smallest positive period r_0 of x is called the *fundamental period* of x. Further if $r \in \mathbb{R}$ is such that $r \cdot x = x$, then there is $z \in \mathbb{Z}$ such that $r = zr_0$.

Definition 2.9. *Let (X, φ) be a flow. A point $x \in X$ is said to be* $\mathbf{r}$-Poison stable *if there is a net $t_\delta \to +\infty$ such that $t_\delta \cdot x \to x$; that is, $x \in \omega^{\mathbf{r}}(x)$. We will denote by $P^{\mathbf{r}}(X)$ the invariant subset of $\mathbf{r}$-Poison stable points of X.*

The reader can easily check that

$$P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X).$$

Definition 2.10. *Let (X, φ) be a flow and $M \subset X$ be an invariant subspace. The subset $A_{\omega^{\mathbf{r}}}(M) = \{x \in X | \omega^{\mathbf{r}}(x) \cap M \neq \emptyset\}$ is called the* region of weak attraction *of M. If $A_{\omega^{\mathbf{r}}}(M)$ is a neighbourhood of M, M is said to be a* weak attractor*, and if we also have that $A_{\omega^{\mathbf{r}}}(M) = X$, M is said to be a* global weak attractor*. The subset $A_{\mathbf{r}}(M) = \{x \in X | \omega^{\mathbf{r}}(x) \neq \emptyset, \omega^{\mathbf{r}}(x) \subset M\}$ is called the* region of attraction *of M. If $A_{\mathbf{r}}(M)$ is a neighbourhood of M, M is said to be an* attractor*, and if we also have that $A_{\mathbf{r}}(M) = X$, M is said to be a* global attractor*.*

The notions above can be dualized to obtain the notion of the $\omega^{\mathbf{l}}$-limit ($\mathbf{l}$ for 'left') set of a point p, the $\Omega^{\mathbf{l}}$- limit of X, $\mathbf{l}$-Poison stable points, repulsors et cetera.

3 The completion of an exterior space

3.1 End Space and Limit Space of an exterior space

In this section we will deal with special limits for a given exterior space. We assume that the reader is familiarized with this particular categorical construction. However, for a definition and main properties of limits (of inverse systems of spaces) we refer the reader to [4]. A more detailed and complete study of some notions given in this subsection can be seen in [14].

Given an exterior space $X = (X, \varepsilon(X))$, its externology $\varepsilon(X)$ can be seen as an inverse system of spaces, then we define the limit space of $(X, \varepsilon(X))$ as the topological space

$$L(X) = \lim \varepsilon(X).$$

Note that for each $E' \in \varepsilon(X)$ the canonical map $\lim \varepsilon(X) \to E'$ is continuous and factorizes as $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E \to E'$. Therefore the canonical map $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E$ is continuous. On the other side, by the universal property of the inverse system, the family of maps $\cap_{E \in \varepsilon(X)} E \to E'$, $E' \in \varepsilon(X)$

induces a continuous map $\cap_{E\in\varepsilon(X)}E \to \lim \varepsilon(X)$. This implies that the canonical map $\lim \varepsilon(X) \to \cap_{E\in\varepsilon(X)}E$ is a homeomorphism.

We recall that for a topological space Y, $\pi_0(Y)$ denotes the set of path-components of Y and we have a canonical map $q_0\colon Y \to \pi_0(Y)$ which induces a quotient topology on $\pi_0(Y)$. We remark that if Y is locally path-connected then $\pi_0(Y)$ is a discrete space.

Definition 3.1. *Given an exterior space $X = (X, \varepsilon(X))$ the* limit space *of X is the topological subspace*

$$L(X) = \lim \varepsilon(X) = \cap_{E\in\varepsilon(X)}E.$$

The end space *of X is the inverse limit*

$$\check{\pi}_0(X) = \lim \pi_0\varepsilon(X) = \lim_{E\in\varepsilon(X)} \pi_0(E)$$

provided with the inverse limit topology of the spaces $\pi_0(E)$.

Note that an end point $a \in \check{\pi}_0(X)$ can be represented by $a = (C_E)_{E\in\varepsilon(X)}$, where C_E is a path-component of E and if $E \subset E'$, $C_E \subset C_{E'}$. In this paper the canonical maps associated to the limit inverse construction will be denoted by $\eta_0\colon \check{\pi}_0(X) \to \pi_0(E)$ (if necessary by $\eta_{0,E}$) given by $\eta_0((C_{E'})_{E'\in\varepsilon(X)}) = C_E$. If $E \subset E'$, we denote by $\eta^E_{E'}\colon \pi_0(E) \to \pi_0(E')$ the natural induced map.

We note that our notion of end point for exterior spaces described by using path-components generalizes the notion of ideal point introduced by Kerékjártó [15] for surfaces and by Freudenthal [7] for more general spaces.

Definition 3.2. *Given an exterior space $X = (X, \varepsilon(X))$, the* bar-limit space *of X is the topological subspace*

$$\bar{L}(X) = \lim_{E\in\varepsilon(X)} \overline{E} = \cap_{E\in\varepsilon(X)}\overline{E}.$$

Remark 3.1. *We note that if an externology $\varepsilon(X)$ satisfies the additional condition that the $\emptyset \notin \varepsilon(X)$, then $\varepsilon(X)$ is a filter of open subsets. The usual notion of the set of cluster points of a filter can also be considered for externologies. In this way, we can say that $\bar{L}(X)$ is the set of cluster points of the externolgy $\varepsilon(X)$.*

We note that $L(X), \bar{L}(X)$ also have natural structures of exterior space with the relative externologies. The relative externology on $L(X)$ is trivial, but this fact is not necessarily true on $\bar{L}(X)$.

It is interesting to observe that if X is an exterior space and X is locally path-connected, $\check{\pi}_0(X)$ is a prodiscrete space. On the other hand, given any exterior space $(X, \varepsilon(X))$, we have a canonical continuous map

$$e_0\colon L(X) \to \check{\pi}_0(X)$$

and a canonical inclusion

$$L(X) \to \bar{L}(X)$$

Definition 3.3. *Given an exterior space $X = (X, \varepsilon(X))$, an end point $a \in \check{\pi}_0(X)$ is said to be* e_0-representable *if there is $p \in L(X)$ such that $e_0(p) = a$. Notice that the map $e_0 \colon L(X) \to \check{\pi}_0(X)$ induce an* e_0-decomposition

$$L(X) = \bigsqcup_{a \in \check{\pi}_0(X)} L_a^0(X)$$

where $L_a^0(X) = e_0^{-1}(a)$. This subset will be called the e_0-component *of the end $a \in \check{\pi}_0(X)$.*

If we denote by $e_0 L(X)$ the subset of representable end points. It is clear that

$$L(X) = \bigsqcup_{a \in e_0 L(X)} L_a^0(X)$$

Example 3.1. *Let $M \colon \mathbb{R} \to (0,1)$ be an increasing continuous map such that $\lim_{t \to -\infty} M(t) = 0$ and $\lim_{t \to +\infty} M(t) = 1$ and take $A = \{e^{2\pi i t} | t \in \mathbb{R}\}$, $B = \{M(t) e^{2\pi i t} | t \in \mathbb{R}\}$. Consider $X = A \cup B \subset \mathbb{C}$ provided with the relative topology (observe that X is not locally connected). On the topological space X the flow $\varphi \colon \mathbb{R} \times X \to X$ is given by $\varphi(r, e^{2\pi i t}) = e^{2\pi i (r+t)}$, $\varphi(r, M(t) e^{2\pi i t}) = M(r+t) e^{2\pi i (r+t)}$. It is clear that this flow has two trajectories A, B. If for each natural number n we denote $B_n = \{M(t) e^{2\pi i t} | t \geq n\}$, we can take the externology $\varepsilon(X) = \{E \in \mathbf{t}_X | \exists n \text{ such that } A \cup B_n \subset E\}$. Since A, B_n are path-connected, it follows that $\pi_0(A \cup B_n) = \{A, B_n\}$. Therefore, one can check that*

$$\check{\pi}_0(X) = \{*_A, *_B\}$$

For this example we have $L(X) = A$, the e_0-decomposition

$$L_{*_A}^0 = A, \quad L_{*_B}^0 = \emptyset$$

This means that $*_B$ is not e_0-representable. Note that $\overline{B_n} = E_n$ is connected. This implies that if we take the set of connected components of an exterior open subset instead of the set of path-components, the corresponding inverse limit will have only one point.

Remark 3.2. *From the definition of e_0-representable end it follows that $e_0 \colon L(X) \to \check{\pi}_0(X)$ is surjective if and only if all end points are e_0-representable.*

Proposition 3.1. *Let X be an exterior space and consider the natural transformation $e_0 \colon L(X) \to \check{\pi}_0(X)$. Then, the following conditions are equivalent:*

(i) $e_0 \colon L(X) \to \check{\pi}_0(X)$ *is injective.*

(ii) *For every $x, x' \in L(X), x \neq x'$, there is $E \in \varepsilon(X)$ and C, C' path-components of E such that $x \in C$, $x' \in C'$ and $C \cap C' = \emptyset$.*

(iii) *For every $x \in L(X)$, $\bigcap_{E \in \varepsilon(X)} C_E(x) = \{x\}$, where $C_E(x)$ denotes the path-component of x in E*

Given an exterior space $X = (X, \varepsilon(X))$ and $E \in \varepsilon(X)$, a subset $W \subset E$ is said to be q_0-saturated if $q_0^{-1}(q_0(W)) = W$; that is, W is a union of path-components of E.

Proposition 3.2. *Let X be an exterior space and suppose that the natural transformation $e_0 \colon L(X) \to \check{\pi}_0(X)$ is a bijection. If for every $x \in L(X)$, $x \in U$, U open in X, there is $E \in \varepsilon(X)$ and W q_0-saturated open subset in E such that $x \in W \subset U$, then $e_0 \colon L(X) \to \check{\pi}_0(X)$ is an open map. Therefore, under these conditions, $e_0 \colon L(X) \to \check{\pi}_0(X)$ is an homeomorphism.*

Proof. Suppose that $x \in L(X)$ and $x \in U$, where U is open in X. By hypothesis, there is $E \in \varepsilon(X)$ and W q_0-saturated open subset in E such that $x \in W \subset U$. Then $e_0(x) \in e_0(W \cap L(X)) \subset e_0(U \cap L(X))$. Since W is q_0-saturated, $e_0(W \cap L(X)) = \eta_0^{-1}(q_0(W))$ is an open subset of $\check{\pi}_0(X)$. □

An interesting property of $\check{\pi}_0(X)$ is given in the next result, whose proof is contained in Theorem 3.17 of [14]. By a locally locally compact at infinity exterior space we mean an exterior space such that for every $E \in \varepsilon(X)$, there is $E' \in \varepsilon(X)$ with $\overline{E'} \subset E$ and $\overline{E'}$ compact.

Proposition 3.3. *Let $X = (X, \varepsilon(X))$ be an exterior space and suppose that X is locally path-connected and locally compact at infinity. Then, $\check{\pi}_0(X)$ is a profinite compact space.*

On the other hand, if X, Y are exterior spaces, and $f \colon X \to Y$ is an exterior map, then f induces continuous maps $L(f) \colon L(X) \to L(Y)$, $\check{\pi}_0(f) \colon \check{\pi}_0(X) \to \check{\pi}_0(Y)$. It is not difficult to check that L preserves exterior homotopies and $\check{\pi}_0$ is invariant by exterior homotopy:

Lemma 3.1. *Suppose that X and Y are exterior spaces and $f, g \colon X \to Y$ exterior maps.*

(i) If $H \colon X \bar{\times} I \to Y$ is an exterior homotopy from f to g, then $L(H) = H|_{L(X) \times I} \colon L(X \bar{\times} I) = L(X) \times I \to L(Y)$ is a homotopy from $L(f)$ to $L(g)$.

(ii) If f is exterior homotopic to g, then $\check{\pi}_0(f) = \check{\pi}_0(g)$.

As a consequence of this lemma one has:

Proposition 3.4. *The functors $L, \check{\pi}_0 \colon \mathbf{E} \to \mathbf{Top}$ induce functors*

$$L \colon \pi\mathbf{E} \to \pi\mathbf{Top}, \qquad \check{\pi}_0 \colon \pi\mathbf{E} \to \mathbf{Top}.$$

3.2 The functor $\check{C}_0 \colon \mathbf{E} \to \mathbf{E}$

In this section, we develop the main construction of this paper: the completion of an exterior space. Later, we will apply this technique to construct some completions of flows.

Given an exterior space $X = (X, \varepsilon(X))$, we can take the following push-out square

$$\begin{tikzcd} L(X) \arrow[r, "e_0"] \arrow[d] & \check{\pi}_0(X) \arrow[d, "\mathrm{in}_0"] \\ X \arrow[r, "p_0"] & X \cup_{L(X)} \check{\pi}_0(X) \end{tikzcd}$$

where $L(X) \to X$ is the canonical inclusion and $p_0 \colon X \to X \cup_{L(X)} \check{\pi}_0(X)$ and $\mathrm{in}_0 \colon \check{\pi}_0(X) \to X \cup_{L(X)} \check{\pi}_0(X)$ are the canonical continuous maps of the push-out.

We can consider the push-out topology: $V \subset X \cup_{L(X)} \check{\pi}_0(X)$ is open if $p_0^{-1}(V)$ is open in X and $\mathrm{in}_0^{-1}(V)$ is open in $\check{\pi}_0(X)$.

Example 3.2. *For $(\mathbb{R}, \mathbf{r})$ we have: $L(X) = \emptyset$, the topology of $X \cup_{L(X)} \check{\pi}_0(X)$ is the disjoint sum of $\mathbb{R}$ and $\{\infty\}$.*

In order to have a good relation between the neighborhoods of an end point and its corresponding family of path-components, we reduce the family of open subsets of the push-out topology to a new family $\mathcal{G}_0$. Recall that for each $E \in \varepsilon(X)$ we have the canonical maps:

$$\begin{tikzcd} & E \arrow[d, "q_0"] \\ \check{\pi}_0(X) \arrow[r, "\eta_0"] & \pi_0(E) \end{tikzcd}$$

Then, given $W \subset X \cup_{L(X)} \check{\pi}_0(X)$, $W \in \mathcal{G}_0$ if and only if it satisfies:

(i) $p_0^{-1}(W)$ is open in X, $\mathrm{in}_0^{-1}(W)$ is open in $\check{\pi}_0(X)$, and

(ii) for each $a \in \mathrm{in}_0^{-1}(W)$ there is $E \in \varepsilon(X)$ and there is G an open in $\pi_0(E)$ such that $a \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W)$, $q_0^{-1}(G) \subset p_0^{-1}(W)$.

Proposition 3.5. *Given an exterior space $(X, \varepsilon(X))$, then the family $\mathcal{G}_0$ of subsets of $X \cup_{L(X)} \check{\pi}_0(X)$ is a topology. Moreover, $in_0 \colon \check{\pi}_0(X) \to in_0(\check{\pi}_0(X))$ is a homeomorphism.*

Proof. Consider a family subsets $W_i \in \mathcal{G}_0$. Since $p_0^{-1}(\cup_i W_i) = \cup_i p_0^{-1}(W_i)$, one has that $p_0^{-1}(\cup_i W_i)$ is open in X and in a similar way $\mathrm{in}_0^{-1}(\cup_i W_i)$ in open in $\check{\pi}_0(X)$.

Suppose that $a \in \mathrm{in}_0^{-1}(W_{i_0})$, then there is $E \in \varepsilon(X)$ and G open in $\pi_0(E)$ such that $a \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W_{i_0}) \subset \mathrm{in}_0^{-1}(\cup_i W_i)$, $q_0^{-1}(G) \subset p_0^{-1}(W_{i_0}) \subset p_0^{-1}(\cup_i W_i)$. Therefore $\cup_i W_i$ is in $\mathcal{G}_0$.

Now suppose that W_1, W_2 are $\mathcal{G}_0$. Since $p_0^{-1}(W_1 \cap W_2) = p_0^{-1}(W_1) \cap p_0^{-1}(W_1)$, one has that $p_0^{-1}(W_1 \cap W_2)$ is open in X and similarly $\mathrm{in}_0^{-1}(W_1 \cap W_2)$ is open in $\check{\pi}_0(X)$. Suppose that $a \in \mathrm{in}_0^{-1}(W_1 \cap W_2)$. Then there are $E_1, E_2 \in \varepsilon(X)$ and G_1 open in $\pi_0(E_1)$, G_2 open in $\pi_0(E_2)$ such that $a \in \eta_0^{-1}(G_1) \subset \mathrm{in}_0^{-1}(W_1)$,

$q_0^{-1}(G_1) \subset p_0^{-1}(W_1)$, $a \in \eta_0^{-1}(G_2) \subset \mathrm{in}_0^{-1}(W_2)$, $q_0^{-1}(G_2) \subset p_0^{-1}(W_2)$. If $E = E_1 \cap E_2 \in \varepsilon(X)$, we can consider the continuous maps $\eta_{E_1}^E \colon \pi_0(E) \to \pi_0(E_1)$, $\eta_{E_2}^E \colon \pi_0(E) \to \pi_0(E_2)$ and $G = (\eta_{E_1}^E)^{-1}(G_1) \cap (\eta_{E_2}^E)^{-1}(G_2)$. This G satisfies $a \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W_1 \cap W_2)$, $q_0^{-1}(G) \subset p_0^{-1}(W_1 \cap W_2)$. Therefore $W_1 \cap W_2$ is in $\mathcal{G}_0$.

We also note that if G open in $\pi_0(E)$, then $\mathrm{in}_0(\eta_0^{-1}(G)) = (p_0(q_0^{-1}(G)) \cup \mathrm{in}_0(\eta_0^{-1}(G))) \cap \mathrm{in}_0(\check{\pi}_0(X))$. This implies that $\mathrm{in}_0 \colon \check{\pi}_0(X) \to \mathrm{in}_0(\check{\pi}_0(X))$ is a homeomorphism.

□

We note that the topology $\mathcal{G}_0$ is coarser than the push-out topology. For instance, one has:

Example 3.3. *For $X = (\mathbb{R}, \mathbf{r})$, $X \cup_{L(X)} \check{\pi}_0(X)$ with the topology $\mathcal{G}_0$ is homeomorphic to $(0, 1]$ with the usual topology.*

Compare this example with the example 3.2. With the new coarser topology $\mathcal{G}_0$, a neighborhood at 1, which corresponds to ∞, always contains a representative path-component of the end point.

If V is a q_0-saturated open subset in E , $E \in \varepsilon(X)$, denote

$$W_0(V) = p_0(V) \cup \mathrm{in}_0(\eta_0^{-1}(q_0(V))).$$

It is easy to check that $p_0^{-1}(W_0(V)) = V$ and $\mathrm{in}_0^{-1}(W_0(V)) = \eta_0^{-1}(q_0(V))$, then by construction $W_0(V) \in \check{C}_0(X)$. As a consequence of this fact, one has:

Lemma 3.2. *If V is a q_0-saturated open subset in E, $E \in \varepsilon(X)$, then $W_0(V)$ is in $\mathcal{G}_0$. In particular, for $V = E$ one has that $W_0(E) = p_0(E) \cup in_0(\check{\pi}_0(X))$ is in $\mathcal{G}_0$ and $p_0^{-1}(W_0(E)) = E$, $in_0^{-1}(W_0(E)) = \check{\pi}_0(X)$.*

We consider the family $\{W_0(E) | E \in \varepsilon(X)\}$. Then if $E_1, E_2 \in \varepsilon(X)$, one can check that $W_0(E_1) \cap W_0(E_2) = W_0(E_1 \cap E_2)$. Now if U is in $\mathcal{G}_0$ and $U \supset W_0(E)$, then $p_0^{-1}(U) \supset p_0^{-1}(W_0(E)) = E$. This implies that $p_0^{-1}(U) \in \varepsilon(X)$ and we have that $W_0(p_0^{-1}(U)) = U$. As a consequence:

Lemma 3.3. *The family $\{W_0(E) | E \in \varepsilon(X)\}$ is an externology in the topological space $(X \cup_{L(X)} \check{\pi}_0(X), \mathcal{G}_0)$.*

Definition 3.4. *The push-out $X \cup_{L(X)} \check{\pi}_0(X)$ with the topology $\mathcal{G}_0$ and the externogy $\{W_0(E) | E \in \varepsilon(X)\}$ has the structure of an exterior space that will be called the* $\check{C}_0$-completion *of X and it will be denoted by $\check{C}_0(X)$.*

In the following examples, we analyze the completion functor for trivial and total externologies.

Example 3.4. *Suppose that $(X, \{X\})$ is a trivial exterior space. Then $L(X) = X$, $\check{\pi}_0(X) = \pi_0(X)$. Therefore, $\check{C}_0(X) = \pi_0(X)$ (notice that we have the quotient topology and the trivial externology).*

Example 3.5. *Suppose that X is a total exterior space $\varepsilon(X) = \mathbf{t}_X$. Then $L(X) = \emptyset$, $\breve{\pi}_0(X) = \emptyset$. Therefore, $\check{C}_0(X) = X$.*

From the properties of the push-out construction one can easily check:

Lemma 3.4. *If X is an exterior space, then $L(\check{C}_0(X)) \cong in_0(\breve{\pi}_0(X))$.*

Proposition 3.6. *The construction $\check{C}_0 \colon \mathbf{E} \to \mathbf{E}$ is a functor and there is a canonical transformation $p_0 \colon \mathrm{id}_{\mathbf{E}} \to \check{C}_0$.*

Proof. Given an exterior map $f \colon X \to Y$, we have to prove that $\check{C}_0(f) \colon \check{C}_0(X) \to \check{C}_0(Y)$ is exterior. Suppose that W is open in $\check{C}_0(Y)$. Then, $p_0^{-1}(W)$ is open in Y, $\mathrm{in}_0^{-1}(W)$ is open in $\breve{\pi}_0(Y)$ and for each $b \in \mathrm{in}_0^{-1}(W)$ there is $E \in \varepsilon(Y)$ and G open in $\pi_0(E)$ such that $b \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W)$, $q_0^{-1}(G) \subset p_0^{-1}(W)$. To see that $(\check{C}_0(f))^{-1}(W)$ is open in $\check{C}_0(X)$, we note that $p_0^{-1}((\check{C}_0(f))^{-1}(W)) = f^{-1}(p_0^{-1}(W))$ is open in X and $\mathrm{in}_0^{-1}((\check{C}_0(f))^{-1}(W)) = (\breve{\pi}_0(f))^{-1}(\mathrm{in}_0^{-1}(W))$ is open in $\breve{\pi}_0(X)$; moreover, for each $a \in \mathrm{in}_0^{-1}((\check{C}_0(f))^{-1}(W))$ there is $E \in \varepsilon(Y)$ and G open in $\pi_0(E)$ such that $\breve{\pi}_0(f)(a) \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W)$, $q_0^{-1}(G) \subset p_0^{-1}(W)$. If we take $f^{-1}(E) \in \varepsilon(X)$ and $(\pi_0(f|_{f^{-1}(E)}))^{-1}(G)$ open in $\pi_0(f^{-1}(E))$, then one has that $a \in \eta_0^{-1}(\pi_0(f|_{f^{-1}(E)})^{-1}(G)) \subset \mathrm{in}_0^{-1}((\check{C}_0(f))^{-1}(W))$ and $q_0^{-1}((\pi_0(f|_{f^{-1}(E)}))^{-1}(G)) \subset p_0^{-1}((\check{C}_0(f))^{-1}(W))$. This implies that $(\check{C}_0(f))^{-1}(W)$ is open in $\check{C}_0(X)$. Therefore $\check{C}_0(f)$ is a continuous map.

To see that $\check{C}_0(f)$ is exterior it suffices to check that $(\check{C}_0(f))^{-1}(W_0(E)) \supset W_0(f^{-1}(E))$ for each $E \in \varepsilon(X)$.

To see that p_0 is a natural transformation, we note that $p_0^{-1}(W_0(E)) = E$. □

Definition 3.5. *An exterior space X is said to be $\check{C}_0$-*complete *if the canonical map $p_0 \colon X \to \check{C}_0(X)$ is an isomorphism in $\mathbf{E}$. An exterior space is said to be $\check{C}_0^2$-*complete *if $\check{C}_0(X)$ is $\check{C}_0$-complete.*

Theorem 3.1. *The functor $\check{C}_0 \colon \mathbf{E}|_{\check{C}_0^2-complete} \to \mathbf{E}|_{\check{C}_0-complete}$ is left adjoint to the inclusion functor $In \colon \mathbf{E}|_{\check{C}_0-complete} \to \mathbf{E}|_{\check{C}_0^2-complete}$.*

Proof. Firstly, we observe that if $X \in \mathbf{E}$ is $\check{C}_0^2$-complete, one has that $\check{C}_0(X)$ is $\check{C}_0$-complete. Now take X $\check{C}_0^2$-complete and Y $\check{C}_0$-complete. If $f \colon \check{C}_0(X) \to Y$ is a map in $\mathbf{E}$, then we have an induced map $fp_0^X \colon X \to Y$. And if $g \colon X \to Y$ is a map in $\mathbf{E}$, then $(p_0^Y)^{-1}\check{C}_0(g) \colon \check{C}_0(X) \to \check{C}_0(Y) \cong Y$ is a map such that $(p_0^Y)^{-1}\check{C}_0(g)p_0^X = g$. The adjuntion above is a consequence of this bijective correspondence. □

Theorem 3.2. *An exterior space X is $\check{C}_0$-complete if and only if the canonical map $L(X) \to \breve{\pi}_0(X)$ is bijective and satisfies the following condition: If $x \in L(X)$, $x \in U$, U open in X, there are $E \in \varepsilon(X)$ and W q_0-saturated open subset in E such that $x \in W \subset U$.*

Proof. We are going to prove that X is $\check{C}_0$-complete by using Proposition 3.2. Firsty, it is easy to check that $p_0 \colon X \to \check{C}_0(X)$ is bijective and exterior. Now,

if U is open in X and $U \cap L(X) = \emptyset$ it is clear that $p_0(U)$ is open in $\check{C}_0(X)$. If $U \cap L(X) \neq \emptyset$, then if $a \in \mathrm{in}_0^{-1}(p_0(U))$, there is a unique $x \in L(X)$ such that $x \in U$ and $p_0(x) = a$. By hypothesis conditions, there are $E \in \varepsilon(X)$ and W q_0-saturated open subset in E such that $x \in W \subset U$. This implies that $q_0(W)$ verifies that $\eta_0^{-1}(q_0(W)) \subset \mathrm{in}_0^{-1} p_0(U)$ and $q_0^{-1} q_0(W) = W \subset U = p_0^{-1} p_0(U)$. Therefore $p_0(U)$ is open in $\check{C}_0(X)$ and p_0 is an exterior homeomorphism. We also remark that if $E \in \varepsilon(X)$, then $p_0(E) = W_0(E)$ is also exterior. This implies that $p_0 \colon X \to \check{C}_0(X)$ is an isomorphism in $\mathbf{E}$. Thus we have obtained that X is $\check{C}_0$-complete. The converse is plain to check. $\square$

Definition 3.6. *An exterior space $X = (X, \varepsilon(X))$ is said to be* first countable at infinity *if $\varepsilon(X)$ has a countable base $E_0 \supset E_1 \supset E_2 \cdots$.*

Note that if an exterior space X is first countable at infinity, then $\check{C}_0(X)$ is first countable at infinity.

Theorem 3.3. *Let X be a locally path-connected exterior space and suppose that X is first countable at infinity. Then, $\check{C}_0(X)$ is $\check{C}_0$-complete.*

Proof. Recall that $L(\check{C}_0(X)) = \bigcap_{E \in \varepsilon(X)} W_0(E) = \mathrm{in}_0(\check{\pi}_0(X))$. Notice that if C is a path-component of E, C is open in X. We consider $W_0(C) = p_0(C) \cup \mathrm{in}_0(\eta_0^{-1}(q_0(C)))$. Let $E_0 \supset E_1 \supset E_2 \cdots$ be a countable base of the externology $\varepsilon(X)$. To prove that there is a path from $x \in p_0(C)$ to $b \in \eta_0^{-1}(q_0(C))$, we can take points $x_n \in E_n \subset E$ and paths α_n from x_n to x_{n+1} to construct an exterior map $\alpha \colon [0, \infty) \to X$ ($[0, \infty)$ with the cocompact externolgy) and an induced map $(p_0 \alpha)' \colon [0, 1] \cong [0, \infty) \cup \{\infty\} \to \check{C}_0(X)$ which is a path from x to b in $W_0(C)$. This implies that $W_0(C)$ is path-connected. Then $\pi_0(E) \to \pi_0(W_0(E))$ verifies that the path-component of $W_0(E)$ that contains C also contains $W_0(C)$ and it is surjective. Since X is locally path-connected we have that $\mathrm{in}_0(\eta_0^{-1}(q_0(C)))$ is open and closed in $\mathrm{in}_0(\check{\pi}_0(X))$. This implies that the path-components of $\mathrm{in}_0(\check{\pi}_0(X))$ are singletons. Thus we obtain that $\pi_0(E) \to \pi_0(W_0(E))$ is injective. Since for every E, $\pi_0(E) \to \pi_0(W_0(E))$ is a bijection, we have that $\check{\pi}_0(X) \to \check{\pi}_0(\check{C}_0(X))$ is bijective. We also have the commutative diagram

$$\begin{tikzcd} L(X) \arrow[r] \arrow[d] & L(\check{C}_0(X)) \arrow[d] \\ \check{\pi}_0(X) \arrow[r] \arrow[ur, "\mathrm{in}_0"] & \check{\pi}_0(\check{C}_0(X)) \end{tikzcd}$$

Since $\check{\pi}_0(X) \to L(\check{C}_0(X))$, and $\check{\pi}_0(X) \to \check{\pi}_0(\check{C}_0(X))$ are bijective, it follows that $L(\check{C}_0(X)) \to \check{\pi}_0(\check{C}_0(X))$ is a continuous bijection. By the definition of topology and externology in $\check{C}_0(X)$ it is easy to check that $\check{C}_0(X)$ satisfies the condition given in Theorem 3.2. Then, it follows that $\check{C}_0(X)$ is $\check{C}_0$-complete.

$\square$

Definition 3.7. *A net x_δ in an exterior space $(X, \varepsilon(X))$ is said to be a* $\varepsilon(X)$-net *if for every $E \in \varepsilon(X)$ there is δ_0 such that for every $\delta \geq \delta_0$, $x_\delta \in E$.*

A net x_δ is said to be a π_0-$\varepsilon(X)$-net *if for every $E \in \varepsilon(X)$ there is a path-component C of E and there is δ_0 such that for every $\delta \geq \delta_0$, $x_\delta \in C$.*

Theorem 3.4. *Let X be a locally path-connected exterior space and for $x \in L(X)$ and $E \in \varepsilon(X)$ denote $C_E(x)$ the path-component of x in E. Then, X is $\check{C}_0$-complete if and only if X satisfies the following conditions:*

(i) *for every $x \in L(X)$ and $U \in (\mathbf{t}_X)_x$, there is $E \in \varepsilon(X)$ such that $C_E(x) \subset U$,*

(ii) *for every $x, y \in L(X)$, $x \neq y$, there is $E \in \varepsilon(X)$ such that $C_E(x) \cap C_E(y) = \emptyset$,*

(iii) *if x_δ is a π_0-$\varepsilon(X)$-net, then there is $x \in L(X)$ such that $x_\delta \to x$.*

Proof. If X is a $\check{C}_0$-complete exterior space, it is easy to check (i),(ii) and (iii). Conversely, if X verifies (i), (ii) and (iii), to prove that X is $\check{C}_0$-complete, we have that condition (i) and the fact that X is locally path-connected imply the corresponding condition given in Theorem 3.2. Then, it suffices to check that the canonical continuous map $e_0 \colon L(X) \to \check{\pi}_0(X)$ is a bijection. Suppose that $a \in \check{\pi}_0(X)$ and $a = \{q_0^{-1}\eta_0^E(a) | E \in \varepsilon(X)\}$. Take $x_E \in q_0^{-1}(\eta_0^E(a))$, then x_E is a π_0-$\varepsilon(X)$-net. By (iii), there is $x \in L(X)$ such that $x_E \to x$. It is easy to check that $e_0(x) = a$. This implies that $e_0 \colon L(X) \to \check{\pi}_0(X)$ is surjective. We can also see that (ii) implies that e_0 is injective.

□

An interesting class of Hausdorff compact $\check{C}_0$-complete spaces are given in the following result:

Theorem 3.5. *Suppose that X is a locally path-connected compact Hausdorff space and $D \subset X$ is a closed totally disconnected subspace. Taking $\varepsilon(X) = \{U | D \subset U, U \in \mathbf{t}_X\}$, then one has X is $\check{C}_0$-complete.*

Proof. In order to apply Theorem 3.4, we are going to check that conditions (i), (ii) and (iii) are satisfied:

(i) If $x \in D \subset U = E$, it is obvious that $C_U(x) \subset U$.

(ii) Under these topological conditions, given an open U such that $D \subset U$, there is an open V such that $D \subset V \subset \mathrm{Cl}(V) \subset U$, where $\mathrm{Cl}(V) = \overline{V}$. Now if $x \in L(X) = D$ one has that $\bigcap_{U \in \varepsilon(X)} C_U(x) = \bigcap_{U \in \varepsilon(X)} \mathrm{Cl}(C_U(x)) \subset D$. Since the inverse limit of continua is a continuum, we have that $\bigcap_{U \in \varepsilon(X)} C_U(x)$ is a connected subset of D. Taking into account that D is totally disconnected, one has that $\bigcap_{U \in \varepsilon(X)} C_0^U(x) = \{x\}$. This implies condition (ii).

(iii) Suppose that x_δ is a π_0-$\varepsilon(X)$-net. From the definition of π_0-$\varepsilon(X)$-net, for each $U \in \varepsilon(X)$ there is a path-componente C_U and δ_U such that $x_\delta \in C_U$ for $\delta \geq \delta_U$. This implies that the family $\{\mathrm{Cl}(C_U)\}$ satisfies the finite intersection property. Since X is compact, we have that $\bigcap_U \mathrm{Cl}(C_U) = \bigcap_U C_U$ is non empty. It is easy to check that if $x \in \bigcap_U C_U \subset D$, we have that $x_\delta \to x$.

□

Theorem 3.6. *Let X be a locally path-connected exterior space.*

(i) If $x, x' \in L(\check{C}_0(X))$, $x \neq x'$, there are open subsets W, W' in $\check{C}_0(X)$ such that $x \in W$, $x' \in W'$ and $W \cap W' = \emptyset$. In particular, we have that $\check{\pi}_0(X) \cong L(\check{C}_0(X))$ is a Hausdorff space.

(ii) If X is a Hausdorff space and for every $x \in X \setminus L(X)$ there is a closed neighborhood F at x such that $X \setminus F$ is exterior, then $\check{C}_0(X)$ is a Hausdorff space. In this case $L(X)$ is a closed subset of X.

Proof. (i) If $x \neq x'$, $x = \mathrm{in}_0(a)$, $x' = \mathrm{in}_0(a')$, then there is $E \in \varepsilon(X)$ such that $\eta_0(a) \neq \eta_0(a')$. If $C = q_0^{-1}(\eta_0(a)), C' = q_0^{-1}(\eta_0(a))$, since X is locally path-connected we have that C, C' are open and $C \cap C' = \emptyset$. This implies that $x \in W_0(C), x' \in W_0(C'), W_0(C) \cap W_0(C') = \emptyset$.

(ii) We can complete the cases analyzed in (i) as follows: If $x \neq x'$ and $\{x, x'\} \cap L(\check{C}_0(X)) = \emptyset$, then $x = p_0(\tilde{x}), x' = p_0(\tilde{x}')$. Now by the hypothesis of (ii) we can construct open subset U, U' of X such that $\tilde{x} \in U$, $\tilde{x}' \in U'$, $U \cap U' = \emptyset$ $(U \cup U') \cap L(X) = \emptyset$. This implies that the open subsets $p_0(U), p_0(U')$ separate x, x'. In the case $x \notin L(\check{C}_0(X))$ and $x' \in L(\check{C}_0(X))$, then $x = p_0(\tilde{x})$ and $\tilde{x} \notin L(X)$. By hypothesis, there is a closed neighborhood F at $\tilde{x}$ such that $X \setminus F$ is exterior. Then, $p_0(\mathrm{int}F)$ and $W_0(X \setminus F)$ separate x, x'. □

Proposition 3.7. *Let X be an exterior space.*

(i) *If x_δ is a π_0-$\varepsilon(X)$-net, then there is $x \in L(\check{C}_0(X))$ such that $p_0(x_\delta) \to x$.*

(ii) *Suppose that X is locally path-connected. If x_δ is a π_0-$\varepsilon(X)$-net, then there is a unique $x \in L(\check{C}_0(X))$ such that $p_0(x_\delta) \to x$.*

Proof. (i) Suppose that x_δ is a π_0-$\varepsilon(X)$-net, then for each $E \in \varepsilon(X)$, there is a path-component C_E of E and δ_E such that for every $\delta \geq \delta_E$, $x_\delta \in C_E$. If is easy to check that if $E' \subset E$, then $C_{E'} \subset C_E$. This implies that $a = (C_E)_{E \in \varepsilon(X)} \in \check{\pi}_0(X)$. Take $x = \mathrm{in}_0(a) \in \mathrm{in}_0(\check{\pi}_0(X)) = L(\check{C}_0(X))$ and suppose that W is a open neighbourhood at x in $\check{C}_0(X)$. Then there is $E \in \varepsilon(X)$ and an open G of $\pi_0(E)$ such that $q_0^{-1}(G) \subset p_0^{-1}(W)$ and $a \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W)$. This implies that $C_E \subset q_0^{-1}(G) \subset p_0^{-1}(W)$, therefore $p_0(x_\delta) \in W$ for every $\delta \geq \delta_E$. Then $p_0(x_\delta) \to x$.

(ii) Now we suppose that that X is locally path-connected and $p_0(x_\delta) \to x$, $p_0(x_\delta) \to x'$, $x, x' \in L(\check{C}_0(X))$. If $x \neq x'$, by (i) of Theorem 3.6, x, x' can be separated by disjoint open subsets W, W'. This fact contradicts that $p_0(x_\delta) \to x$, $p_0(x_\delta) \to x'$. □

Remark 3.3. *There are $\check{C}_0$-complete locally path-connected exterior spaces X having a π_0-$\varepsilon(X)$-net x_δ such that there is $y \in X \setminus L(X)$ such that $x_\delta \to y$. Nevertheless, there is a unique $x \in L(X)$ with $x_\delta \to x$.*

Remark 3.4. *It is remarkable that if X is a $\check{C}_0$-complete exterior space, one has that $L(X)$ is a "weak attractor" of X in the sense that every π_0-$\varepsilon(X)$-net x_δ has a limit point in $L(X)$. If X is also locally path-connected, then x_δ*

may have different limit points, but there is a unique limit point in $L(X)$. If, in addition, X is also Hausdorff, then $L(X)$ is closed and every π_0-$\varepsilon(X)$-net has a unique limit point in X. Then $L(X)$ is an "attractor" in the sense that every π_0-$\varepsilon(X)$-net x_δ has a unique limit point and this limit point is in $L(X)$. In this case, since $L(X) = L(\check{C}_0(X)) \cong \check{\pi}_0(X)$, one has that $L(X)$ is a Hausdorff, totally disconected space (a prodiscrete space).

Next we analyze the compactness properties of completions:

Theorem 3.7. *Let X be an exterior space. Suppose that for every $E \in \varepsilon(X)$, there is $E' \in \varepsilon(X)$ such that $E' \subset E$ and the image of $\pi_0(E') \to \pi_0(E)$ is finite. Then,*

(i) $L(\check{C}_0(X))$ is compact,

(ii) Denote $\eta_{0,E}(\check{\pi}_0(X))$ the image of $\eta_{0,E} \colon \check{\pi}_0(X) \to \pi_0(E)$. If for every $E \in \varepsilon(X)$, $X \setminus (\bigcup_{C \in \eta_{0,E}(\check{\pi}_0(X))} C)$ is compact, then $\check{C}_0(X)$ is compact.

Proof. (i) The hypothesis condition implies that $\check{\pi}_0(X)$ is a profinite space (i.e., an inverse limit of finite discrete spaces). Since a profinite space is compact and $L(\check{C}_0(X)) = \mathrm{in}_0(\check{\pi}_0(X))$, we have that $L(\check{C}_0(X))$ is compact.

(ii) Take an open covering $\{W_i | i \in I\}$ of $\check{C}_0(X)$. For each $x \in L(\check{C}_0(X))$, there is W_{i_x} such that $x \in W_{i_x}$. By the definition of the topology $\mathcal{G}_0$, there is $E_x \in \varepsilon(X)$ and G_x open in $\pi_0(E_x)$ such that $x \in W_0(q_0^{-1}(G_x)) \subset W_{i_x}$. Since $L(\check{C}_0(X))$ is compact, there is a finite set $\{x_1, \cdots x_k\}$ such that $L(\check{C}_0(X)) \subset W_0(q_0^{-1}(G_{x_1})) \cup \cdots \cup W_0(q_0^{-1}(G_{x_k}))$. Take $E = E_{x_1} \cap \cdots \cap E_{x_k} \in \varepsilon(X)$ and G'_{x_i} the inverse image of G_{x_i} via the map $\pi_0(E) \to \pi_0(E_{x_i})$. Then, one has

$$L(\check{C}_0(X)) \subset W_0(q_0^{-1}(G'_{x_1})) \cup \cdots \cup W_0(q_0^{-1}(G'_{x_k})) \subset W_{x_1} \cup \cdots \cup W_{x_k}.$$

Since by hypothesis $X \setminus (q_0^{-1}(G'_{x_1}) \cup \cdots \cup q_0^{-1}(G'_{x_k}))$ is a closed compact subset of X we have that $p_0(X \setminus (q_0^{-1}(G'_{x_1}) \cup \cdots \cup q_0^{-1}(G'_{x_k})))$ is compact. Taking into account that

$$\check{C}_0(X) = (W_0(q_0^{-1}(G'_{x_1})) \cup \cdots \cup W_0(q_0^{-1}(G'_{x_k}))) \cup p_0(X \setminus (q_0^{-1}(G'_{x_1}) \cup \cdots \cup q_0^{-1}(G'_{x_k})))$$

the finite family $\{W_{i_{x_1}}, \cdots, W_{i_{x_k}}\}$ together with a finite family covering of $p_0(X \setminus (q_0^{-1}(G'_{x_1}) \cup \cdots \cup q_0^{-1}(G'_{x_k})))$ give a finite subcovering of the space $\check{C}_0(X)$. □

Theorem 3.8. *Suppose that X is a locally path-connected, connected, Hausdorff exterior space and $\varepsilon(X) \subset \varepsilon^{\mathbf{c}}(X)$. If for every $x \in X \setminus L(X)$ there is a closed neighbourhood F such that $X \setminus F$ is exterior, then $\check{C}_0(X)$ is a Hausdorff compact space and $L(X)$ is a closed subspace.*

Proof. Firstly we see, that under these conditions we also have that if K is a closed compact subset of X contained in $X \setminus L(X)$, then $X \setminus K \in \varepsilon(X)$. Indeed, for each $x \in K$ there is a closed compact neighborhood F_x at x such

that $X \setminus F_x \in \varepsilon(X)$. Since K is compact, there is a finite $\{x_1, \cdots, x_k\}$ such that $K \subset F_{x_1} \cup \cdots \cup F_{x_k}$. This implies that $X \setminus K \supset X \setminus F_{x_1} \cap \cdots \cap X \setminus F_{x_k}$ and therefore $X \setminus K \in \varepsilon(X)$. Notice that we also have proved that if K is a closed compact subset of $X \setminus L(X)$, there is a closed compact $K' = F_{x_1} \cup \cdots \cup F_{x_k}$ $(X \setminus K' \in \varepsilon(X))$ such that $K \subset \mathrm{int}(K')$.

Now, given $E_1 \in \varepsilon(X)$, we are going to prove that there is $E_2 \in \varepsilon(X)$ such that $\mathrm{Cl}(E_2) \subset \mathrm{int}(E_1)$. Notice that the condition $\varepsilon(X) \subset \varepsilon^{\mathbf{c}}(X)$ implies that the frontier $\mathrm{Fr}(E_1)$ is a closed compact contained in $X \setminus L(X)$. By the argument above, there is K' such that $\mathrm{Fr}(E_1) \subset \mathrm{int}(K')$ and taking $E_2 = E_1 \setminus K' \in \varepsilon(X)$, we have that $\mathrm{Cl}(E_2) \subset E_1$.

Suppose that C_2 is a path-component of E_2 and C_1 the path-component of E_1, such that $C_2 \subset C_1$. If $C_1 \cap \mathrm{Fr}(E_2) = \emptyset$, consider $x_1 \in C_1$ and a path $\alpha\colon [0,1] \to C_1 \subset E_1$ such that $\alpha(0) = x_1$, $\alpha(1) = x_2 \in C_2$. Note that $\alpha^{-1}(E_2) = \alpha^{-1}(\mathrm{Cl}(E_2))$ is a non-empty open and closed subset of $[0,1]$. This implies that $\alpha^{-1}(E_2) = [0,1]$; that is, $\alpha([0,1]) \subset E_2$. Thus one has that $x_1 \in C_2$ and $C_1 = C_2$. Since C_1 is a closed subset of E_1 and $\mathrm{Cl}(E_2) \subset E_1$, one has that C_1 is an open and closed subset of X, contradicting the fact that X is connected. Then we have that $C_1 \cap \mathrm{Fr}(E_2) \neq \emptyset$. Since $\mathrm{Fr}(E_2) \subset \bigcup_{C \in \pi_0(E_1)} C$ and $\mathrm{Fr}(E_2)$ is compact, we obtain that there is a finite number of components having non empty intersection with $\mathrm{Fr}(E_2)$. This implies that the image of $\pi_0(E_2) \to \pi_0(E_1)$ is finite.

We note that $\mathrm{Im}(\check{\pi}_0(X) \to \pi_0(E_1)) \subset \mathrm{Im}(\pi_0(E_2) \to \pi_0(E_1))$ is a finite set.

One the other hand, suppose that $E \in \varepsilon(X)$ and $C \in \pi_0(E)$. It is easy to check that $C \in \mathrm{Im}(\check{\pi}_0(X) \to \pi_0(E))$ if and only if for every $E' \in \varepsilon(X)$ such that $E' \subset E$, $C \cap E' \neq \emptyset$. Using this characterization, the fact that $\varepsilon(X)$ is closed by finite intersections and the images above are finite, it follows that there exists $E_3 \in \varepsilon(X)$ such that $E_3 \subset E_1$, $\mathrm{Im}(\check{\pi}_0(X) \to \pi_0(E_1)) = \mathrm{Im}(\pi_0(E_3) \to \pi_0(E_1))$ and $E_3 \subset \bigcup_{C \in \eta_{0,E_1}(\check{\pi}_0(X))} C$. Since $X \setminus E_3$ is compact and $\bigcup_{C \in \eta_{0,E_1}(\check{\pi}_0(X))} C$ is open, we have that $X \setminus (\bigcup_{C \in \eta_{0,E_1}(\check{\pi}_0(X))} C)$ is compact.

Now applying Theorems 3.6 and 3.7 one has that $\check{C}_0(X)$ is a Hausdorff compact space and $L(\check{C}_0(X))$ is a closed subspace.

$\square$

Corollary 3.1. *Suppose that X is a locally compact, locally path-connected, connected, Hausdorff space and $\varepsilon(X) = \varepsilon^{\mathbf{c}}(X)$. Then $L(X) = \emptyset$, $\check{\pi}_0(X) = \mathcal{F}(X)$ is the space of Freudenthal ends of X and $\check{C}_0(X)$ is a Hausdorff compact space.*

Remark 3.5. *We point out that, with the hypothesis of the corollary above, the underlying topological space of $\check{C}_0(X)$ is exactly the Freudenthal compactification of X, see [7].*

4 The category of r-exterior flows

In this section we are going to consider the exterior space $\mathbb{R}^{\mathbf{r}} = (\mathbb{R}, \mathbf{r})$. We recall (see subsection 2.1) that $\mathbf{r}$ is the following externology:

$$\mathbf{r} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (n, +\infty) \subset U\}.$$

The exterior space $\mathbb{R}^{\mathbf{r}}$ plays an important role in the definition of $\mathbf{r}$-*exterior flow* below. Such notion mixes the structures of dynamical system and exterior space (see [6],[14]):

Definition 4.1. *Let M be an exterior space, $M_{\mathbf{t}}$ denote the subjacent topological space and $M_{\mathbf{d}}$ denote the set M provided with the discrete topology. An* $\mathbf{r}$-exterior flow *is a continuous flow $\varphi\colon \mathbb{R}\times M_{\mathbf{t}} \to M_{\mathbf{t}}$ such that $\varphi\colon \mathbb{R}^{\mathbf{r}} \bar{\times} M_{\mathbf{d}} \to M$ is exterior and for any $t \in \mathbb{R}$, $F_t\colon M \bar{\times} I \to M$, $F_t(x,s) = \varphi(ts, x)$, $s \in I$, $x \in M$, is also exterior.*

An $\mathbf{r}$-exterior flow morphism *of $\mathbf{r}$-exterior flows is a flow morphism $f\colon M \to N$ such that f is exterior. We will denote by $\mathbf{E^rF}$ the category of $\mathbf{r}$-exterior flows and $\mathbf{r}$-exterior flow morphisms.*

Given an $\mathbf{r}$-exterior flow $(M, \varphi) \in \mathbf{E^rF}$, one also has a flow $(M_{\mathbf{t}}, \varphi) \in \mathbf{F}$. This gives a forgetful functor

$$(\cdot)_{\mathbf{t}}\colon \mathbf{E^rF} \to \mathbf{F}.$$

Now given a continuous flow $X = (X, \varphi)$, an open $N \in \mathbf{t}_X$ is said to be $\mathbf{r}$-*exterior* if for any $x \in X$ there is $T^x \in \mathbf{r}$ such that $\varphi(T^x \times \{x\}) \subset N$. It is easy to check that the family of $\mathbf{r}$-exterior subsets of X is an externology, denoted by $\varepsilon^{\mathbf{r}}(X)$, which gives an exterior space $X^{\mathbf{r}} = (X, \varepsilon^{\mathbf{r}}(X))$ such that $\varphi\colon \mathbb{R}^{\mathbf{r}} \bar{\times} X_{\mathbf{d}} \to X^{\mathbf{r}}$ is exterior and $F_t\colon X^{\mathbf{r}} \bar{\times} I \to X^{\mathbf{r}}$, $F_t(x,s) = \varphi(ts, x)$, is also exterior for every $t \in \mathbb{R}$. Therefore $(X^{\mathbf{r}}, \varphi)$ is an $\mathbf{r}$-exterior flow which is said to be the $\mathbf{r}$-*exterior flow associated to* X. When there is no possibility of confusion, $(X^{\mathbf{r}}, \varphi)$ will be briefly denoted by $X^{\mathbf{r}}$. Then we have a functor

$$(\cdot)^{\mathbf{r}}\colon \mathbf{F} \to \mathbf{E^rF}.$$

The category of flows can be considered as a full subcategory of the category of exterior flows:

Proposition 4.1. *The functor $(\cdot)^{\mathbf{r}}\colon \mathbf{F} \to \mathbf{E^rF}$ is left adjoint to the functor $(\cdot)_{\mathbf{t}}\colon \mathbf{E^rF} \to \mathbf{F}$. Moreover $(\cdot)_{\mathbf{t}}\,(\cdot)^{\mathbf{r}} = \mathrm{id}$ and $\mathbf{F}$ can be considered as a full subcategory of $\mathbf{E^rF}$ via $(\cdot)^{\mathbf{r}}$.*

4.1 End Spaces and Limit Spaces of an exterior flow

In section 3.1 we have defined the end and limit spaces of an exterior space. In particular, since any $\mathbf{r}$-exterior flow X is an exterior space, we can consider the end space $\breve{\pi}_0(X)$ and the limit space $L(X)$. Notice that one has the following properties:

Proposition 4.2. *Suppose that $X = (X, \varphi)$ is an $\mathbf{r}$-exterior flow. Then*

(i) The space $L(X)$ is invariant.

(ii) There is a trivial flow structure induced on $\check{\pi}_0(X)$.

Proof. (i) We have that $L(X) = \cap_{E\in\varepsilon(X)} E$. Note that for any $s \in \mathbb{R}$, $\varphi_s(E) \in \varepsilon(X)$ if and only if $E \in \varepsilon(X)$. Then $\varphi_s(L(X)) = \varphi_s(\cap_{E\in\varepsilon(X)} E) = \cap_{E\in\varepsilon(X)} \varphi_s(E) = \cap_{E\in\varepsilon(X)} E = L(X)$.

(ii) For any $s \in \mathbb{R}$, consider the exterior homotopy $F_s \colon X \bar{\times} I \to X$, $F_s(x,t) = \varphi(ts, x)$, from id_X to φ_s. By Lemma 3.1, it follows that $\mathrm{id} = \check{\pi}_0(\varphi_s)$.

□

As a consequence of this result, one has functors $L, \check{\pi}_0 \colon \mathbf{E^rF} \to \mathbf{F}$.

Proposition 4.3. *The functors $L, \check{\pi}_0 \colon \mathbf{E^rF} \to \mathbf{F}$ induce functors*

$$L \colon \pi\mathbf{E^rF} \to \pi\mathbf{F}, \quad \check{\pi}_0 \colon \pi\mathbf{E^rF} \to \mathbf{F},$$

where the homotopy categories are constructed in a canonical way.

4.2 The end point of a trajectory and the induced decompositions of an exterior flow

For an $\mathbf{r}$-exterior flow X, one has that each trajectory has an end point given as follows: Given $p \in X$ and $E \in \varepsilon(X)$, there is $T^p \in \mathbf{r}$ such that $T^p \cdot p \subset E$. We can suppose that T^p is path-connected and therefore so is $T^p \cdot p$; this way there is a unique path-component $\omega^0_{\mathbf{r}}(p, E)$ of E such that $T^p \cdot p \subset \omega^0_{\mathbf{r}}(p, E) \subset E$. This gives maps $\omega^0_{\mathbf{r}}(\cdot, E) \colon X \to \pi_0(E)$ and $\omega^0_{\mathbf{r}} \colon X \to \check{\pi}_0(X)$ such that the following diagram commute:

$$\begin{tikzcd} L(X) \arrow[r, "e_0"] \arrow[d] & \check{\pi}_0(X) \\ X \arrow[ur, "\omega^0_{\mathbf{r}}"] & \end{tikzcd}$$

These maps permit to divide a flow in simpler subflows.

Definition 4.2. *Let X be an $\mathbf{r}$-exterior flow. We will consider $X^0_{(\mathbf{r},a)} = (\omega^0_{\mathbf{r}})^{-1}(a)$, $a \in \check{\pi}_0(X)$. The invariant space $X^0_{(\mathbf{r},a)}$ will be called the $\omega^0_{\mathbf{r}}$-*basin *at $a \in \check{\pi}_0(X)$.*

The map $\omega^0_{\mathbf{r}}$ induces the following partition of X in simpler flows

$$X = \bigsqcup_{a\in\check{\pi}_0(X)} X^0_{(\mathbf{r},a)}$$

*that will be called respectively, the $\omega^0_{\mathbf{r}}$-*decomposition *of the $\mathbf{r}$-exterior flow X.*

It is important to note that the map $\omega^0_{\mathbf{r}}$ needs not be continuous.

Definition 4.3. *Let X be an $\mathbf{r}$-exterior flow. An end point $a \in \check{\pi}_0(X)$ is said to be $\omega^0_{\mathbf{r}}$-*representable *if there is $p \in X$ such that $\omega^0_{\mathbf{r}}(p) = a$. Denote by $\omega^0_{\mathbf{r}}(X)$ the space of $\omega^0_{\mathbf{r}}$-representable end points.*

Since the $\omega^0_{\mathbf{r}}$-decomposition of X is compatible with the e_0-decomposition of the limit subspace, we have a canonical map $e_0L(X) \to \omega^0_{\mathbf{r}}(X)$.

5 The completion of an r-exterior flow

In this section, we use the completion functor $\check{C}_0 \colon \mathbf{E} \to \mathbf{E}$ given in subsection 3.2 to construct the completion functor $\check{C}_0 \colon \mathbf{E^rF} \to \mathbf{E^rF}$.

Given an $\mathbf{r}$-exterior flow $X = (X, \varphi)$, we can take the following diagram in the category of topological spaces **Top**, where the front and back faces are push-outs in **Top**

$$\begin{tikzcd}
& L(X) \arrow[rr, "e_0"] \arrow[dd] & & \check{\pi}_0(X) \arrow[dd, "\mathrm{in}_0"] \\
\mathbb{R} \times L(X) \arrow[ur, "\varphi|_{\mathbb{R}\times L(X)}"] \arrow[rr, "\mathrm{id}\times e_0"] \arrow[dd] & & \mathbb{R} \times \check{\pi}_0(X) \arrow[ur, "\mathrm{pr}"] \arrow[dd] & \\
& X \arrow[rr, "p_0"] & & X \cup_{L(X)} \check{\pi}_0(X) \\
\mathbb{R} \times X \arrow[ur, "\varphi"] \arrow[rr, "\mathrm{id}\times p_0"] & & \mathbb{R} \times (X \cup_{L(X)} \check{\pi}_0(X)) \arrow[ur, dotted, "\check{\varphi}_0"] &
\end{tikzcd}$$

Therefore, considering such push-out topologies, there is an induced continuous map

$$\check{\varphi}_0 \colon \mathbb{R} \times (X \cup_{L(X)} \check{\pi}_0(X)) \to X \cup_{L(X)} \check{\pi}_0(X)$$

In order to prove that $\check{\varphi}_0$ is continuous with the topology $\mathcal{G}_0$ we introduce some notation and study some properties:

Given $S \subset X$ and $t \in \mathbb{R}$, denote by

$$S^t = \{q \in S | st \cdot q \in S, \ \ 0 \leq s \leq 1\}$$

We note that if $S_1 \subset S_2$, then $S_1^t \subset S_2^t$.

Lemma 5.1. *Suppose that U is an open in a flow X and $t \in \mathbb{R}$. Then,*

(i) U^t is an open subset of X,

(ii) if A is a path-component of U^t, B is a path-component of U and $A \subset B$, then $A \subset B^t$.

Proof. (i) Suppose that $q \in U^t$; then for each $0 \leq s \leq 1$ there are $W(s) \in (\mathbf{t}_{[0,1]})_s$ and $V(s) \in (\mathbf{t}_X)_q$ such that $W(s)t \cdot V(s) \subset U$. Since $[0,1]$ is compact, we can find $V \in (\mathbf{t}_X)_q$ such that for all $0 \leq s' \leq 1$, $s't \cdot V \subset U$. This implies that $V \subset U^t$. Therefore U^t is open. (ii) Take $q \in A$, then $q \in ([0,1]t) \cdot q \subset U$ and $([0,1]t) \cdot q$ is path-connected, then we have that $([0,1]t) \cdot q \subset B$. This implies that $A \subset B^t$. $\square$

Lemma 5.2. *Suppose that $f \colon X \to Y$ is a morphism of $\mathbb{R}$-sets. If $B \subset Y$, then $f^{-1}(B^t) = (f^{-1}(B))^t$.*

Proof. Just take into account that $f(st \cdot x) = st \cdot f(x)$. $\square$

Proposition 5.1. *Let $X = (X, \varphi)$ be an $\mathbf{r}$-exterior flow. Then, $\check{\varphi}_0$ induces on $\check{C}_0(X)$ an $\mathbf{r}$-exterior flow structure; that is:*

(i) *the map $\check{\varphi}_0 \colon \mathbb{R} \times \check{C}_0(X) \to \check{C}_0(X)$ is continuous (with the topology $\mathcal{G}_0$)*

(ii) *the map $\check{\varphi}_0 \colon \mathbb{R}^{\mathbf{r}} \bar{\times} \check{C}_0(X)_{\mathbf{d}} \to \check{C}_0(X)$ is exterior, and*

(iii) *for any $t \in \mathbb{R}$, $F_t \colon \check{C}_0(X) \bar{\times} I \to \check{C}_0(X)$, $F_t(x, s) = \check{\varphi}_0(ts, x)$, $s \in I$, $x \in \check{C}_0(X)$, is also exterior.*

Proof. (i) Firstly, recall that if $f \colon X \to Y$ is an exterior map, by Proposition 3.6, $\check{C}_0(f)$ is a continuous map (in fact is an exterior map). In particular, given $X = (X, \varphi)$ an $\mathbf{r}$-exterior flow, since φ_t is an exterior map and $(\check{\varphi}_0)_t = \check{C}_0(\varphi_t)$, one has that $(\check{\varphi}_0)_t$ is a continuous map.

We observe that if Y is a topological space and there is a set action $\phi \colon \mathbb{R} \times Y \to Y$ such that each ϕ_t is continuous and ϕ is continuous on $\{0\} \times Y$, then ϕ is continuous. Therefore, in our case, it suffices to check that the action $\check{\varphi}_0 \colon \mathbb{R} \times \check{C}_0(X) \to \check{C}_0(X)$ is continuous on $\{0\} \times \check{C}_0(X)$; that is, if $0 \cdot y = y \in W$ with W an open subset of $\check{C}_0(X)$, then there exist $\delta > 0$ and an open W' containing y such that $[-\delta, \delta] \cdot W' \subset W$.

Before proving the continuity we analyze some previous properties:

Suppose that $W \subset \check{C}_0(X)$ is open and take $t \in \mathbb{R}$. By Lemma 5.2, $(p_0^{-1}(W))^t = p_0^{-1}(W^t)$. We also have that $\mathrm{in}_0^{-1}(W^t) = (\mathrm{in}_0^{-1}(W))^t = \mathrm{in}_0^{-1}(W)$ since the action on $\check{\pi}_0(X)$ is trivial.

We know that $\mathrm{in}_0^{-1}(W)$ is open in $\check{\pi}_0(X)$ and for each $a \in \mathrm{in}_0^{-1}(W^t) = \mathrm{in}_0^{-1}(W)$ there is $E \in \varepsilon(X)$ and G open in $\pi_0(E)$ such that $a \in \eta_0^{-1}(G) \subset \mathrm{in}_0^{-1}(W)$, $q_0^{-1}(G) \subset p_0^{-1}(W)$. Taking E^t and $(\eta_E^{E^t})^{-1}G$, one has that $a \in \eta_0^{-1}((\eta_E^{E^t})^{-1}G) \subset \mathrm{in}_0^{-1}(W^t)$, $q_0^{-1}((\eta_E^{E^t})^{-1}G) \subset (p_0^{-1}(W))^t$. This implies that W^t is also open. Now, taking the definition of W^t, one has that for any $t > 0$ $[-t, t] \cdot (W^t \cap W^{-t}) \subset W$.

Now to prove that $\check{\varphi}_0$ is continuous at $(0, y)$ we distinguish two cases:

a) Suppose that $y \in W \subset \check{C}_0(X)$ with $y = p_0(x)$ and $x \in p_0^{-1}(W)$. Then, there are $\varepsilon > 0$ and $U \in (\mathbf{t}_X)_x$ such that $[-\varepsilon, \varepsilon] \cdot U \subset p_0^{-1}(W)$. This implies that $y \in W^{\varepsilon} \cap W^{-\epsilon}$ and therefore $[-\varepsilon, \varepsilon] \cdot (W^{\varepsilon} \cap W^{-\varepsilon}) \subset W$.

b) Otherwise, $y \notin p_0(X)$, so $y \in \mathrm{in}_0(\check{\pi}_0(X))$ (y is not e_0-representable). In this case we have that for any $t > 0$, $y \in W^t$ and $[-t, t] \cdot (W^t \cap W^{-t}) \subset W$.

The proof of (ii) and (iii) is a routine checking.

□

Definition 5.1. *Given an* $\mathbf{r}$*-exterior flow* $X = (X, \varphi)$*, the* $\mathbf{r}$*-exterior flow* $\check{C}_0(X) = (\check{C}_0(X), \check{\varphi}_0)$ *is called the* $\check{C}_0$-completion *of* X*.*

Taking into account that for all $E \in \varepsilon(X)$, $p_0(E) \subset W_0(E)$ the following result holds.

Proposition 5.2. *The construction* $\check{C}_0 \colon \mathbf{E^rF} \to \mathbf{E^rF}$ *is a functor and the canonical map* $p_0 : X \to \check{C}_0(X)$ *induces a natural transformation.*

Notice that $\mathbb{R}$ with the usual order is a directed set. Then, if X is a flow, $\varphi^x \colon \mathbb{R} \to X$ is a net of the space X, for each $x \in X$.

Theorem 5.1. *Let* $X \in \mathbf{E^rF}$ *be an* $\mathbf{r}$*-exterior flow. Then,*

(i) *For any* $x \in X$*,* φ^x *is a* π_0*-*$\varepsilon(X)$*-net.*

(ii) *For any* $x \in X$*,* $p_0\varphi^x \to \omega_r^0(x)$ *in* $\check{C}_0(X)$ *(*$\omega_r^0(x)$ *is identified to* $in_0(\omega_r^0(x))$*).*

(iii) *If* X *is locally path-connected and* $p_0\varphi^x \to a$*,* $p_0\varphi^x \to b$*,* $a, b \in \check{\pi}_0(X) \subset \check{C}_0(X)$*, then* $a = b$*.*

Proof. (i) Given $E \in \varepsilon(X)$, since φ^x is exterior, there is $n_E \in \mathbb{R}$ such that $\varphi^x([n_E, \infty)) \subset E$. Since $[n_E, \infty)$ is path-connected, there is a path-component C_E of E such that $\varphi^x([n_E, \infty)) \subset C_E$. This implies that φ^x is a π_0-$\varepsilon(X)$-net.

(ii) Notice that $\omega_r^0(x) = (C_E)_{E\in\varepsilon(X)}$. If $\mathrm{in}_0(\omega_r^0(x)) \in W$ and W is open, there is $E \in \varepsilon(X)$ and G open in $\pi_0(E)$ such that $q_0^{-1}(G) \subset p_0^{-1}(W)$ and $\omega_r^0(x) \in \eta_{0,E}^{-1}(G) \subset \mathrm{in}_0^{-1}(W)$. Then $\varphi^x([n_E, \infty)) \subset C_E \subset q_0^{-1}(G) \subset p_0^{-1}(W)$ and therefore $p_0\varphi^x([n_E, \infty)) \subset W$.

(iii) Since X is locally path-connected, by Theorem 3.6 we have that $\check{\pi}_0(X)$ is a Hausdorff space. This implies that $p_0\varphi^x$ has a unique limit within $\mathrm{in}_0(\check{\pi}_0(X)) = L(\check{C}_0(X))$.

□

If we apply the theorem above to a $\check{C}_0$-complete $\mathbf{r}$-exterior flow, we obtain the next result.

Theorem 5.2. *Let* $X \in \mathbf{E^rF}$ *be a* $\check{C}_0$*-complete* $\mathbf{r}$*-exterior flow. Then,*

(i) *For any* $x \in X$*,* $\varphi^x \to e_0^{-1}\omega_{\mathbf{r}}^0(x)$ *and therefore* $e_0^{-1}\omega_{\mathbf{r}}^0(x) \in \omega^{\mathbf{r}}(x)$*.*

(ii) *If* X *is Hausdorff, then* $\omega^{\mathbf{r}}(x) = \{e_0^{-1}\omega_{\mathbf{r}}^0(x)\}$ *and* $C(X) = P(X) = \overline{\Omega^{\mathbf{r}}(X)}$*.*

Proof. (i) By Theorem 5.1, $p_0\varphi^x \to \mathrm{in}_0(\omega_r^0(x))$, so $\varphi^x \to p_0^{-1}\mathrm{in}_0(\omega_r^0(x)) = e_0^{-1}(\omega_r^0(x))$.

(ii) Using the convergence properties of nets and subnets and the fact that X is Hausdorff, one has $\omega^{\mathbf{r}}(x) = \{e_0^{-1}\omega_{\mathbf{r}}^0(x)\}$. Since $\omega^{\mathbf{r}}(x)$ is invariant, it follows that $\Omega^{\mathbf{r}}(X) \subset C(X)$. Obviously, one has that $C(X) \subset P(X) \subset \Omega^{\mathbf{r}}(X)$. Therefore, it follows that $C(X) = P(X) = \Omega^{\mathbf{r}}(X)$. Taking into account that

set of critical points in a Hausdorff flow is closed, it is not difficult to check that $C(X) = P(X) = \overline{\Omega^{\mathbf{r}}(X)}$.

$\square$

Taking into account the definition of global weak attractor and global attractor, see Definition 2.10, the following is obtained:

Corollary 5.1. *Let $X \in \mathbf{E^rF}$ be a $\check{C}_0$-complete $\mathbf{r}$-exterior flow. Then,*

(i) *$L(X)$ is a global weak attractor of X and $L(X) = C(X)$.*

(ii) *If X is Hausdorff, then $L(X)$ is the unique minimal global attractor of X and $L(X) = \overline{\Omega^{\mathbf{r}}(X)}$.*

Proof. (i) By (i) of Theorem 5.2, we have that for every $x \in X$, $e_0^{-1}\omega_{\mathbf{r}}^0(x) \in \omega^{\mathbf{r}}(x)$, so $\omega^{\mathbf{r}}(x) \cap L(X) \neq \emptyset$. Thus, $L(X)$ is a global weak attractor. It is clear that $C(X) \subset L(X)$. Since X is $\check{C}_0$-complete, one has that $L(X) \cong \check{\pi}_0(X)$ as flows. Taking into account that the action on $\check{\pi}_0(X)$ is trivial, it follows that every point of $L(X)$ is critical. We conclude $L(X) = C(X)$.

(ii) Assume that $M \subset L(X)$ is a global weak attractor. If $x \in L(X) \setminus M$, since X is T_1 and x is critical, we have that

$$\omega^{\mathbf{r}}(x) = \bigcap_{t \geq 0} \overline{[t, +\infty) \cdot x} = \overline{\{x\}} = \{x\}.$$

This contradicts the minimality of M. On the other hand, we have that $L(X) = C(X) \subset P(X) \subset \Omega^{\mathbf{r}}(X)$. But by Theorem 5.2, $\omega^{\mathbf{r}}(x) = \{\omega_{\mathbf{r}}^0(x)\} \subset L(X)$, implying that $L(X) = C(X) = \Omega^{\mathbf{r}}(X)$. Finally, since in a Hausdorff flow the set of critical points $C(X)$ is closed, one has that $L(X) = \overline{\Omega^{\mathbf{r}}(X)}$.

$\square$

6 Ends, Limits and completions of a flow via exterior flows

Recall that we have considered the functor

$$(\cdot)^{\mathbf{r}} \colon \mathbf{F} \to \mathbf{E^rF},$$

the forgetful functor

$$(\cdot)_{\mathbf{t}} \colon \mathbf{E^rF} \to \mathbf{F},$$

the functors

$$L, \check{\pi}_0 \colon \mathbf{E^rF} \to \mathbf{F},$$

and

$$\bar{L}, \check{C}_0 \colon \mathbf{E^rF} \to \mathbf{E^rF}.$$

Therefore we can consider the composites:

$$L^{\mathbf{r}} := L(\cdot)^{\mathbf{r}}, \;\; \check{\pi}_0^{\mathbf{r}} := \check{\pi}_0(\cdot)^{\mathbf{r}}, \;\; \bar{L}^{\mathbf{r}} := (\cdot)_{\mathbf{t}}\bar{L}(\cdot)^{\mathbf{r}}, \;\; \check{C}_0^{\mathbf{r}} := (\cdot)_{\mathbf{t}}\check{C}_0(\cdot)^{\mathbf{r}}$$

to obtain functors $L^{\mathbf{r}}$, $\check{\pi}_0^{\mathbf{r}}, \bar{L}^{\mathbf{r}}, \check{C}_0^{\mathbf{r}} \colon \mathbf{F} \to \mathbf{F}$.

In this way, given a flow X, we have the *end (trivial) flow* $\check{\pi}_0^{\mathbf{r}}(X) = \check{\pi}_0(X^{\mathbf{r}})$, the *limit flow* $L^{\mathbf{r}}(X) = L(X^{\mathbf{r}})$, the *bar-limit flow* $\bar{L}^{\mathbf{r}}(X) = \bar{L}(X^{\mathbf{r}})$ and the *completion flow* $\check{C}_0^{\mathbf{r}}(X) = (\check{C}_0(X^{\mathbf{r}}))_{\mathbf{t}}$.

It is interesting to consider the following equivalence of categories: Given any flow $\varphi \colon \mathbb{R} \times X \to X$, one can consider the *reversed flow* $\varphi' \colon \mathbb{R} \times X \to X$ defined by $\varphi'(r, x) = \varphi(-r, x)$, for every $(r, x) \in \mathbb{R} \times X$. The correspondence $(X, \varphi) \to (X, \varphi')$ gives rise to a functor

$$(\cdot)' \colon \mathbf{F} \to \mathbf{F}$$

which is an equivalence of categories and verifies $(\cdot)'(\cdot)' = \mathrm{id}$. Using the composites

$$L^{\mathbf{l}} := (\cdot)' L^{\mathbf{r}}(\cdot)', \;\; \check{\pi}_0^{\mathbf{l}} := (\cdot)' \check{\pi}_0^{\mathbf{r}}(\cdot)', \;\; \bar{L}^{\mathbf{l}} := (\cdot)' \bar{L}^{\mathbf{r}}(\cdot)', \;\; \check{C}_0^{\mathbf{l}} := (\cdot)' \check{C}_0^{\mathbf{r}}(\cdot)',$$

we obtain the new functors $L^{\mathbf{l}}, \check{\pi}_0^{\mathbf{l}}, \bar{L}^{\mathbf{l}}, \check{C}_0^{\mathbf{l}} \colon \mathbf{F} \to \mathbf{F}$.

Definition 6.1. *A flow X is said to be* $\check{C}_0^{\mathbf{r}}$-complete *(*$\check{C}_0^{\mathbf{l}}$-complete*) if the canonical map $p_0 \colon X \to \check{C}_0^{\mathbf{r}}(X)$ is an homeomorphism. A flow is said* $(\check{C}_0^{\mathbf{r}})^2$-complete ($(\check{C}_0^{\mathbf{l}})^2$-complete*) if $\check{C}_0^{\mathbf{r}}(X)$ ($\check{C}_0^{\mathbf{l}}(X)$) is $\check{C}_0^{\mathbf{r}}$-complete ($\check{C}_0^{\mathbf{l}}$-complete).*

We remark that in this definition we denote $(p_0)_{\mathbf{t}}$ by p_0. In fact, for a flow X, one has that $p_0 \colon X^{\mathbf{r}} \to \check{C}_0(X^{\mathbf{r}})$ is an isomorphism in $\mathbf{E}^{\mathbf{r}}\mathbf{F}$ if and only if $p_0 = (p_0)_{\mathbf{t}} \colon X \to \check{C}_0^{\mathbf{r}}(X)$ is a flow homeomorphism in $\mathbf{F}$.

For the following results we will use some previous properties whose proofs are given in Theorem 6.3, Lemma 6.11 and Corollary 6.14 of [14], respectively.

Proposition 6.1. *Let X be a flow and suppose that X is a T_1-space. Then*

$$P(X) = L^{\mathbf{r}}(X).$$

Proposition 6.2. *Let X be a flow and suppose that X is a locally compact regular space. If $x \notin \overline{\Omega^{\mathbf{r}}(X)}$, then there exists $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V}_x$ is* $\mathbf{r}$-*exterior.*

Proposition 6.3. *Let X be a flow. If X is a locally compact T_3 space, then $L^{\mathbf{r}}(X) = P(X)$, $\bar{L}^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$ and*

$$L^{\mathbf{r}}(X) = P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X) \subset \overline{\Omega^{\mathbf{r}}(X)} = \bar{L}^{\mathbf{r}}(X).$$

Then the following result holds:

Theorem 6.1. *Let X be a $\check{C}_0^{\mathbf{r}}$-complete flow. Then,*

(i) *$L^{\mathbf{r}}(X)$ is a global weak attractor of X and $L^{\mathbf{r}}(X) = C(X)$.*

(ii) *If X is T_1, then $L^{\mathbf{r}}(X)$ is a minimal global weak attractor of X and $L^{\mathbf{r}}(X) = P(X)$.*

(iii) *If X is T_2, then $L^{\mathbf{r}}(X)$ is the unique minimal global attractor of X and $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$*

(iv) *If X is locally compact and T_3, then $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(\check{C}_0(X))} = \bar{L}^{\mathbf{r}}(X)$.*

Proof. (i) It follows from (i) of Corollary 5.1.

(ii) Suppose that $M \subset L^{\mathbf{r}}(X)$ is a global weak attractor. Take $x \in L^{\mathbf{r}}(X)$; since X is T_1 and x is critical, we have that $\omega^{\mathbf{r}}(x) = \{x\}$ and therefore $x \in M$. By Proposition 6.1 one has that $L^{\mathbf{r}}(X) = P(X)$.

(iii) follows from (ii) of Corollary 5.1 and (iv) is a consequence of Proposition 6.3. □

Theorem 6.2. *Let X be a locally path-connected, locally compact T_2 flow. Then, $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$ if and only if $\check{C}_0^{\mathbf{r}}(X)$ is a T_2 flow.*

Proof. Ir order to apply Theorem 3.6, take $x \in X \setminus L^{\mathbf{r}}(X)$. Since $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$ by Theorem 6.1, we have that $x \notin \overline{\Omega^{\mathbf{r}}(X)}$. By Proposition 6.2, there is $V_x \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V_x}$ is $\mathbf{r}$-exterior. It follows that $\check{C}_0^{\mathbf{r}}(X)$ is a T_2 flow.

Conversely, suppose that $\check{C}_0^{\mathbf{r}}(X) = \check{C}_0(X^{\mathbf{r}})$ is a T_2 flow. By Theorem 6.1, $L(\check{C}_0(X^{\mathbf{r}})) = P(\check{C}_0(X^{\mathbf{r}})$. We also have that $C(\check{C}_0(X^{\mathbf{r}})) \subset P(\check{C}_0(X^{\mathbf{r}})$ and in this case, since $P(\check{C}_0(X^{\mathbf{r}}) = L(\check{C}_0(X^{\mathbf{r}})) \cong \check{\pi}_0(X)$ has a trivial action, we have that $P(\check{C}_0(X^{\mathbf{r}}) \subset C(\check{C}_0(X))$. Therefore $L(\check{C}_0(X^{\mathbf{r}})) = P(\check{C}_0(X^{\mathbf{r}}) = C(\check{C}_0(X^{\mathbf{r}}))$.

We also have that $C(\check{C}_0^{\mathbf{r}}(X))$ is a closed space because $\check{C}_0^{\mathbf{r}}(X)$ is T_2 . This implies that

$$L^{\mathbf{r}}(X) = p_0^{-1}(L(\check{C}_0^{\mathbf{r}}(X))) = p_0^{-1}(C(\check{C}_0^{\mathbf{r}}(X)))$$

is also closed. Now suppose $\Omega^{\mathbf{r}}(X) \setminus L^{\mathbf{r}}(X) \neq \emptyset$, then there are φ^x and $y \in \Omega^{\mathbf{r}}(X) \setminus L(X^{\mathbf{r}})$ such that $\varphi^x(t_\delta) \to y$, $t_\delta \to +\infty$. This implies that $p_0(\varphi^x(t_\delta)) \to p_0(y)$ and $p_0(\varphi^x(t_\delta)) \to \omega_{\mathbf{r}}^0(x)$ and $\omega_{\mathbf{r}}^0(x) \neq p_0(y)$. Then $\check{C}_0^{\mathbf{r}}(X)$ is not T_2, which is a contradiction. Therefore, $L^{\mathbf{r}}(X) = \underline{\Omega^{\mathbf{r}}(X)}$. Finaly, taking into account that $L^{\mathbf{r}}(X)$ is closed we have that $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$. □

Definition 6.2. *A flow X is said to be* Lagrange stable *if for every $x \in X$ the semi-trajectory $[0, \infty) \cdot x$ is contained in a compact subspace.*

Corollary 6.1. *Let X be a locally path-connected, locally compact T_2 flow and suppose that X is Lagrange stable. Then, $L^{\mathbf{r}}(X)$ is the minimal closed global attractor if and only if $\check{C}_0^{\mathbf{r}}(X)$ is a T_2 flow.*

Proof. Since X is Lagrange stable, we have that for every $x \in X$, $\omega^{\mathbf{r}}(x) \neq \emptyset$. If M is a closed global attractor, then $\Omega^r(X) \subset M$ and $\overline{\Omega^r(X)} \subset \overline{M} = M$. Therefore the unique minimal global closed attractor of X is $\overline{\Omega^r(X)}$. Now it suffices to apply the result given in Theorem 6.2. □

Theorem 6.3. *Let X be a locally path-connected, locally compact T_2 flow satisfying that $X^{\mathbf{r}}$ is first countable at infinity. Then, $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$ if and only if $\check{C}_0^{\mathbf{r}}(X)$ is a T_2 $\check{C}_0^{\mathbf{r}}$-complete flow.*

Proof. It follows from Theorem 6.2 and Theorem 3.3. □

Theorem 6.4. *Let X be a locally compact, locally path-connected, connected T_2 flow and $\varepsilon(X^{\mathbf{r}}) \subset \varepsilon^{\mathbf{c}}(X)$. Then, $L^{\mathbf{r}}(X) = P(X) = \overline{\Omega^{\mathbf{r}}(X)}$ if and only if $\check{C}_0^{\mathbf{r}}(X)$ is a T_4 compact flow.*

Proof. It is a consequence of Proposition 6.2, Theorem 3.8 and Theorem 6.2. □

Theorem 6.5. *Let X be a second countable, locally compact, locally path-connected, connected T_2 flow and $\varepsilon(X^{\mathbf{r}}) \subset \varepsilon^{\mathbf{c}}(X)$. Then, $L^{\mathbf{r}}(X) = P(X) = \overline{\Omega^{\mathbf{r}}(X)}$ if and only if $\check{C}_0^{\mathbf{r}}(X)$ is a T_4 compact $\check{C}_0^{\mathbf{r}}$-complete flow.*

Proof. Suppose $L^{\mathbf{r}}(X) = P(X) = \overline{\Omega^{\mathbf{r}}(X)}$. Taking into account that $X \setminus L^{\mathbf{r}}(X)$ is also second countable and Proposition 6.2, one can find an increasing sequence K_n of compacts with $X \setminus K_n \in \varepsilon(X^{\mathbf{r}})$ which covers $X \setminus L^{\mathbf{r}}(X)$. We have that $X \setminus K_n$ is an exterior countable base of $\varepsilon(X^{\mathbf{r}})$. Now, we just have to use Theorem 6.4 and Theorem 6.3. □

Corollary 6.2. *Let X be a locally path-connected, connected, compact metric flow. Then, $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$ ($L^{\mathbf{r}}(X)$ is the minimal global attractor) if and only if $\check{C}_0^{\mathbf{r}}(X)$ is a T_2 compact $\check{C}_0^{\mathbf{r}}$-complete flow.*

Proof. Note that for a compact space, T_2 is equivalent to T_4, and for a metric space, second contable is equivalent to Lindelöf. We also have that a compact space is a Lindelöf space. Therefore, under these hypothesis we can apply Theorem 6.5. □

Definition 6.3. *A topological space X is said to be a* Stone space *if X is Hausdorff, compact and totally disconnected.*

Lemma 6.1. *Let X be a T_2 compact flow. If $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$, then $\varepsilon(X^{\mathbf{r}}) = \{U | L^{\mathbf{r}}(X) \subset U, U \in \mathbf{t}_X\}$.*

Proof. We always have that $\varepsilon^{\mathbf{r}}(X) \subset \{U | L^{\mathbf{r}}(X) \subset U, U \in \mathbf{t}_X\}$. In our case, since $L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$, by Proposition 6.2, we have that if K is a closed compact subset and $K \cap L^{\mathbf{r}}(X) = \emptyset$, then $X \setminus K \in \varepsilon^{\mathbf{r}}(X)$. Taking into account that X is compact, if $L^{\mathbf{r}}(X) \subset U \in \mathbf{t}_X$, then $X \setminus U$ is compact and therefore $U \in \varepsilon^{\mathbf{r}}(X)$. □

Theorem 6.6. *Let X be a locally path-connected, compact metric flow. Then, X is a $\check{C}_0^{\mathbf{r}}$-complete flow if and only if $C(X) = \overline{\Omega^{\mathbf{r}}(X)}$ (i.e. $C(X)$ is the minimal global attractor) and $C(X)$ is a Stone space.*

Proof. Suppose that X is a $\check{C}_0^{\mathbf{r}}$-complete. By Theorems 6.2 and 6.1, it follows that $C(X) = L^{\mathbf{r}}(X) = \overline{\Omega^{\mathbf{r}}(X)}$, and by Lemma 6.1 $\varepsilon(X^{\mathbf{r}}) = \{U | L(X) \subset U, U \in \mathbf{t}_X\}$. Now since X is T_4, one has that $X^{\mathbf{r}}$ is locally compact at infinity, that is, for any $E \in \varepsilon(X^{\mathbf{r}})$ there is $E' \in \varepsilon(X^{\mathbf{r}})$ such that $\overline{E'} \subset E$. Applying Proposition 3.3, we obtain that $C(X) = L^{\mathbf{r}}(X) \cong \check{\pi}_0^{\mathbf{r}}(X)$ is a profinite compact space; that is, a Stone space. The converse follows from Lemma 6.1 and Theorem 3.5. □

Remark 6.1. *We also have a similar version of the results above simply by using the functors* $\check{C}_0^{\mathbf{l}}, L^{\mathbf{l}}, \bar{L}^{\mathbf{l}}$ *and the notion of repulsor.*

Example 6.1. *For a Morse function [17]* $f\colon M \to \mathbb{R}$*, where* M *is a compact* T_2 *Riemannian manifold, one has that the opposite of the gradient of* f *induces a flow with a finite number of critical points. In this case, we have that* M *is locally path-connected and the flow is* $\mathbf{r}$*-locally compact at infinity. Then we have that* $L^{\mathbf{r}}(X) = C(X) = L^{\mathbf{l}}(X)$ *is a finite set and* X *is a* $\check{C}_0^{\mathbf{r}}$*-complete and* $\check{C}_0^{\mathbf{l}}$*-complete flow.*

7 More completion functors for flows

The authors think that it could be interesting to complete the study presented in this paper with other posible completions. We suggest to work with $\mathbf{r}$-exterior flows and the following functors:

a) The functor $\check{W}_0\colon \mathbf{E^rF} \to \mathbf{E^rF}$.

Let $(X, \varepsilon(X))$ be an exterior flow. Notice that we have the following natural transformation $\omega_{\mathbf{r}}^0(X) \to \check{\pi}_0(X)$. Taking the push-out square

$$\begin{tikzcd}
L(X) \arrow[r, "e_0"] \arrow[d] & \omega_{\mathbf{r}}^0(X) \arrow[d, "\mathrm{in}_0"] \\
X \arrow[r, "p_0"] & X \cup_{L(X)} \omega_{\mathbf{r}}^0(X)
\end{tikzcd}$$

and taking similar topologies and externologies we obtain the completion $\check{W}_0(X)$ and a canonical transformation $\check{W}_0(X) \to \check{C}_0(X)$.

The authors think that the study of this new completion and the possible relations between $\check{W}_0$-completions and $\check{C}_0$-completions will give interesting properties and results.

b) The functor $\check{C}\colon \mathbf{E^rF} \to \mathbf{E^rF}$.

Given an exterior space $(X, \varepsilon(X))$, for every $E \in \varepsilon(X)$ one can take the set of connected componets of E instead of the set of path-connected components of E. This gives a different notion of end point and the corresponding completion functor. Using these new constructions some of the above results can be reformulated removing the condition of being first countable at infinity.

c) The functors $\check{\check{C}}_0, \check{\check{C}}\colon \mathbf{E^rF} \to \mathbf{E^rF}$.

In these cases, for an exterior space $(X, \varepsilon(X))$ and for every $E \in \varepsilon(X)$ one can take the set of connected componets of $\overline{E}$ and the set of path-connected components of $\overline{E}$. We can take the bar-limit functor and the correspondings end points to construct new completion functors in order to obtain new results about global attractors.

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